Research Statement

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1 Introduction

The theory of integer partitions is classical and well developed, but is still an active area of research with a large number of open questions. Indeed, this basic notion of counting ways to write natural numbers n as sums of smaller natural numbers introduces a bevy of questions whose answers, both full and partial, have required tools such as modular forms, probability, and the Hardy-Littlewood method.

Explicitly, *partitions* of $n \in \mathbb{N}$ are tuples of positive integers $(a_1, a_2, \ldots, a_k)$ such that $a_1 \geq a_2 \geq \ldots \geq a_k$ and $a_1 + a_2 + \cdots + a_k = n$. Using $\Pi[n]$ to denote the set of such partitions, the quantities $\mathfrak{p}(n) := \mathrm{card}(\Pi[n])$ are the (ordinary) *partition numbers*. Modern partition theory essentially originated in 1918 with Hardy and Ramanujan's famous asymptotic formulae for $\mathfrak{p}(n)$ as $n \to \infty$. Namely, in [14] they establish (in addition to stronger results) that

$$\log \mathfrak{p}(n) \sim \kappa\sqrt{n}, \qquad \text{where } \kappa := \pi\sqrt{2/3} \tag{1.1}$$

and the relation $a(n) \sim b(n)$ indicates that $\lim_{n\to\infty} a(n)/b(n) = 1$. At nearly the same time, the striking "Ramanujan congruences" were established. Namely, for all $n \geq 0$

$$\mathfrak{p}(5n+4) \equiv 0 \pmod 5, \quad \mathfrak{p}(7n+5) \equiv 0 \pmod 7, \quad \text{and} \quad \mathfrak{p}(11n+6) \equiv 0 \pmod{11}.$$

Recent years have seen a surge of interest in partition theory led by researchers including B. Berndt, A. Malik, R. C. Vaughan, and A. Zaharescu (see, e.g., [2, 9, 22, 23]). Here we discuss our work on both arithmetic and analytic aspects of the novel class of "signed" partition enumerations involving multiplicative $f : \mathbb{N} \to \{0, \pm 1\}$. Following this we discuss a few problems toward developing more general results on some families of signed partition numbers.

2 Past Work

Let $f : \mathbb{N} \to \{0, \pm 1\}$, and for $n \in \mathbb{N}$ and any partition $\pi = (a_1, a_2, \ldots, a_k) \in \Pi[n]$ let

$$f(\pi) := f(a_1)f(a_2)\cdots f(a_k).$$

With this we define the *signed partition numbers*

$$\mathfrak{p}(n, f) = \sum_{\pi \in \Pi[n]} f(\pi). \tag{2.1}$$

Definition (2.1) generalizes several classical partition-related quantities, e.g., with the constant function 1 one has $\mathfrak{p}(n) = \mathfrak{p}(n, 1)$, and with the indicator function $\mathbf{1}_A$ for $A \subset \mathbb{N}$, the quantities $\mathfrak{p}(n, \mathbf{1}_A)$ are the *A-restricted partition numbers*. Many examples and families of restricted partitions numbers are well studied (see, e.g., [1, 11]).

2.1 Arithmetic work; q-series and periodic vanishings

A sequence $(a_n)_{\mathbb{N}}$ *vanishes on an arithmetic progression* (or has a *periodic vanishing*) if one has $a_{mj+r} = 0$ for some $m \in \mathbb{N}$, some $0 \leq r < m$, and all $j \geq 0$. In such a case we may say that a_n "vanishes for $n \equiv r \pmod{m}$". The past two decades have seen a boom of work on periodic vanishings in the coefficients of various q-series; we note [3, 4, 17, 18, 20] as only a small portion of the extant literature.

Very recently I discovered and established two 10-periodic vanishings in sequences $(\mathfrak{p}(n, f))_{\mathbb{N}}$ related to the Legendre symbol $\chi_5(n) := (\frac{n}{5})$; namely $\chi_5(n)$ is 1 (or -1) when n is (or is not) a quadratic residue modulo 5, and $\chi_5(n)$ is 0 when $5 \mid n$. In addition, for $n \in \mathbb{N}$ and any partition $\pi = (a_1, a_2, \ldots, a_k)$ of n let

$$\chi_5^{\dagger}(\pi) = (-1)^k \chi_5(a_1)\chi_5(a_2)\cdots\chi_5(a_k).$$

The primary result of [8] establishes the following 10-periodic vanishings in $(\mathfrak{p}(n, \chi_5))_{\mathbb{N}}$ and $(\mathfrak{p}(n, \chi_5^{\dagger}))_{\mathbb{N}}$.

Theorem 2.1. *One has* $\mathfrak{p}(10j + 2, \chi_5) = 0$ *and* $\mathfrak{p}(10j + 6, \chi_5^{\dagger}) = 0$ *for all* $j \geq 0$.

Two further relations on the above sequences are also demonstrated in [8].

Theorem 2.2. *One has* $\mathfrak{p}(10j, \chi_5^{\dagger}) = \mathfrak{p}(10j, \chi_5)$ *and* $\mathfrak{p}(10j + 8, \chi_5^{\dagger}) = -\mathfrak{p}(10j + 8, \chi_5)$ *for all* $j \geq 0$.

In [8], Theorems 2.1 and 2.2 are proven using the theory of *q-series* identities and extensive symbolic manipulations. To assist any readers, a Mathematica notebook further documenting and implementing these computations is available on our personal webpage. A heuristic explanation of the periodic vanishing demonstrated by the sequence $(\mathfrak{p}(n, \chi_5))_{\mathbb{N}}$ is provided by the following asymptotic formula for $\mathfrak{p}(n, \chi_5)$.

Theorem 2.3 ([7, Thm. 1.7])**.** *As* $n \to \infty$ *one has*

$$\mathfrak{p}(n, \chi_5) = \mathfrak{a}_5 n^{-3/4} \exp\Big(\tfrac{1}{2}\kappa\sqrt{\tfrac{4}{5}n}\Big)\big[1 + (-1)^n \mathfrak{b}_5 + \mathfrak{d}_5 \cos\big(\tfrac{2\pi}{5}n - \tfrac{\pi}{10}\big) + O(n^{-1/5})\big], \tag{2.2}$$

where

$$\kappa = \pi\sqrt{\tfrac{2}{3}}, \quad \mathfrak{a}_5 = \left(\frac{3+\sqrt{5}}{960}\right)^{1/4}, \quad \mathfrak{b}_5 = \frac{3-\sqrt{5}}{2}, \quad \textit{and} \quad \mathfrak{d}_5 = \sqrt{2(5-\sqrt{5})}.$$

In particular, ignoring the error term $O(n^{-1/5})$ in (2.2) and considering the 10-periodic "signed" term

$$\mathfrak{S}(n) := 1 + (-1)^n\left(\frac{3-\sqrt{5}}{2}\right) + \sqrt{2(5-\sqrt{5})}\,\cos\big(\tfrac{2\pi}{5}n - \tfrac{\pi}{10}\big),$$

it is surprising to find that

$$\mathfrak{S}(2) = 0 \qquad \text{and} \qquad \mathfrak{S}(n) \neq 0 \quad \text{for } 1 \leq n \leq 10 \text{ with } n \neq 2.$$

This provides a soft explanation for the periodic vanishing seen in $(\mathfrak{p}(n, \chi_5))_{\mathbb{N}}$. The surprising nature of this periodic vanishing is amplified by the following further result.

Theorem 2.4 ([7, Thm. 1.10])**.** *For odd primes* p*, let* $\chi_p(n)$ *denote the Legendre symbol* $(\frac{n}{p})$*. If* $p \neq 5$ *and* $p \not\equiv 1 \pmod{8}$*, then the sequence* $(\mathfrak{p}(n, \chi_p))_{\mathbb{N}}$ *does not vanish on any arithmetic progression.*

2.2 Asymptotic results on some $\mathfrak{p}(n, f)$

When f assumes both positive and negative values one expects these signs to cause cancellations in the sums $\mathfrak{p}(n, f)$. We recall the Möbius μ and Liouville λ functions from prime number theory: If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ with distinct primes p_i and all $a_i \geq 1$, then

$$\lambda(n) := (-1)^{a_1+\cdots+a_r} \qquad \text{and} \qquad \mu(n) := \begin{cases} (-1)^r & \text{if all } a_i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The following result is an immediate corollary of the main results of [6].

Theorem 2.5. *For all $\varepsilon > 0$, as $n \to \infty$ one has*

$$\mathfrak{p}(n, \mu) = O\big(e^{(1+\varepsilon)\sqrt{n}}\big) \qquad \textit{and} \qquad \mathfrak{p}(n, \lambda) = O\big(e^{(\frac{1}{2}+\varepsilon)\kappa\sqrt{n}}\big), \tag{2.3}$$

where $\kappa = \pi\sqrt{2/3}$. In addition, for positive integer k, as $k \to \infty$ one has

$$\log \mathfrak{p}(2k, \mu) \sim \sqrt{2k} \qquad \textit{and} \qquad \log \mathfrak{p}(2k, \lambda) \sim \tfrac{1}{2}\kappa\sqrt{2k}. \tag{2.4}$$

Given the relations of (2.4), it is natural to consider to what extent those relations "extend" to odd n. In [5] this question is answered under mild assumptions on the zeros of the Riemann zeta function $\zeta(s)$. Let $\Theta := \sup\{\mathrm{Re}(\rho) : \zeta(\rho) = 0\}$. It is well known that $\frac{1}{2} \leq \Theta \leq 1$; the assertion that $\Theta = \frac{1}{2}$ is the Riemann Hypothesis (RH).

Again for odd primes p let $\chi_p(n)$ denote the Legendre symbol $(\frac{n}{p})$. In [7], asymptotic formulae for different families of *Legendre-signed partition numbers* $\mathfrak{p}(n, \chi_p)$ are established, where primes are separated by their residue modulo 8. Such formulae are largely similar in structure to the formula for $\mathfrak{p}(n, \chi_5)$ in Theorem 2.3, but the general constants $\mathfrak{a}_p$ and $\mathfrak{b}_p$ corresponding to $\mathfrak{a}_5$ and $\mathfrak{b}_5$ often have unwieldy formulae. Thus, here we only include the asymptotic formulae for $p \equiv 1 \pmod 4$ and $p \neq 5$.

Theorem 2.6 ([7, Thm. 1.3])**.** *Let p be an odd prime such that $p \neq 5$ and $p \equiv 1 \pmod 4$, and let $L(s, \chi_p)$ be the Dirichlet L-function for χ_p. As $n \to \infty$ one has*

$$\mathfrak{p}(n, \chi_p) = \mathfrak{a}_p n^{-3/4} \exp\Big(\tfrac{1}{2}\kappa\sqrt{(1-\tfrac{1}{p})n}\Big)\big[1 + (-1)^n \mathfrak{b}_p + O(n^{-1/5})\big],$$

where

$$\kappa = \pi\sqrt{2/3}, \qquad \mathfrak{a}_p = 2^{-7/4} 3^{-1/4} (p^{-1} - p^{-2})^{1/4} \exp(\tfrac{1}{4}\sqrt{p} L(1, \chi_p))$$

and

$$\mathfrak{b}_p = \begin{cases} 1 & p \equiv 1 \pmod 8, \\ \exp(-\sqrt{p} L(1, \chi_p)) & p \equiv 5 \pmod 8 \textit{ and } p \neq 5. \end{cases}$$

3 Proposed research

The novelty of the results of Theorems 2.3 and 2.4 indicate that sequences $\mathfrak{p}(n, \chi_p)_{\mathbb{N}}$ involving primes $p \equiv 1 \pmod 8$ should be further explored. Basic empirical computations have been done, but these computations further indicate that the periodic vanishing of $\mathfrak{p}(n, \chi_5)$ is indeed quite rare.

Problem 3.1. Establish the presence of periodic vanishings, or lack thereof, in sequences $(\mathfrak{p}(n,\chi_p))_{\mathbb{N}}$ where $p \equiv 1 \pmod 8$.

A function $f : \mathbb{N} \to \{\pm 1\}$ is *totally multiplicative* if $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{N}$; clearly such f are completely determined by their values $f(p)$ on primes. Among totally multiplicative f, the constant function 1 and the Liouville function λ are the "most positive" and "most negative", respectively, since one has $1(p) = 1$ and $\lambda(p) = -1$ for all primes p. The relation (1.1) and Theorem 2.5 show that the "exponential factor" for $\mathfrak{p}(n,\lambda)$ is, in essence, one half of that for $\mathfrak{p}(n,1)$. It is thus natural to investigate the following problem.

Problem 3.2. Suppose that $f : \mathbb{N} \to \{\pm 1\}$ is totally multiplicative, and for all real x let $\log_0 x := \max\{\log x, 0\}$. Determine if

$$\tfrac{1}{2} \le \limsup_{n\to\infty} \left| \frac{\log_0 \mathfrak{p}(n,f)}{\log \mathfrak{p}(n,1)} \right| \le 1. \tag{3.1}$$

We briefly describe how sequences $(\mathfrak{p}(n,f))_{\mathbb{N}}$ are often analyzed using the Hardy-Littlewood method. First, for $f : \mathbb{N} \to \{0, \pm 1\}$ one defines

$$\Phi(z,f) = \prod_{n=1}^{\infty} (1 - f(n)z^n)^{-1} \quad \text{and} \quad \Psi(z,f) = \sum_{k,n=1}^{\infty} \frac{f^k(n)}{k} z^{nk} \quad (|z| < 1). \tag{3.2}$$

In addition, let $e(\alpha) := \exp(2\pi i\alpha)$. By Cauchy's theorem, for $\rho \in (0,1)$ one has

$$\mathfrak{p}(n,f) = \frac{1}{2\pi i} \int_{|z|=\rho} \Phi(z) z^{-n-1}\, dz = \rho^{-n} \int_0^1 \Phi(\rho e(\alpha)) e(-n\alpha)\, d\alpha. \tag{3.3}$$

As $\Phi(z) = \exp \Psi(z)$, integrals (3.3) are analyzed via $\Psi(z)$ rather than $\Phi(z)$. Specifically, one considers $\Psi(\rho e(\alpha), f)$ when α is in different connected subsets of $[0,1)$, namely the *arcs* of the Hardy-Littlewood method. The *major arcs* are intervals centered on reduced rationals $a/q \in [0,1]$ with denominator bounded by some chosen $Q \ge 1$, and the *minor* arcs are the connected components of the $[0,1)$-complement of the major arcs. One may specially designate the major arcs about 0 and 1 the *principal* arcs.

In [12], Gafni describes information on a general $A \subset \mathbb{N}$ necessary for a successful application of the Hardy-Littlewood method to the integrals (3.3). Here we give a similar description of three sums involving $f(n)$ used to apply the Hardy-Littlewood method to analyze $\Psi(\rho e(\alpha), f)$. Specifically, one requires good estimates on: (1) the Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$; (2) sums $\sum_{n\le x} f(nq+r)$ where $0 \le r < q$; and (3) sums $\sum_{n\le x} f(n)e(n\alpha)$. Estimates on these three sums facilitate the analyses of $\Psi_f(\rho e(\alpha))$ for α in the principal, major, and minor arcs, respectively.

In analyzing different $(\mathfrak{p}(n,f))_{\mathbb{N}}$, the series $\sum_{n=1}^{\infty} f(n)n^{-s}$ and the integrals (3.3) for α in the principal arcs are often the easiest of the three arcs to analyze, and a number of results on sequences $(\mathfrak{p}(n,f))_{\mathbb{N}}$ with certain general Dirichlet series already exist, e.g., [19,21]. In [7] we establish the following lemma concerning a general family of $\Psi_f(\rho e(\alpha))$ with α in certain minor arcs of $[0,1)$.

Lemma 3.3. *Let f be multiplicative with $|f| \leq 1$, let $X > 0$ be sufficiently large, and let $\alpha \in [0,1)$ have the property that: If $(a,q) = 1$ and $|q\alpha - a| \leq X^{-2/3}$, then $q > X^{1/3}$. Then*

$$\Psi(\rho e(\alpha), f) \ll X/\log X.$$

Results on sums $\sum_{n \leq x} f(n)e(n\alpha)$ with multiplicative f and α near to small denominator a/q have been recently established in a work of de la Bretèche and Granville [10] (building on work of Granville, Harper, and Soundararajan [13]). Although these results are not immediately useable for results on general $\Psi_f(\rho e(\alpha))$, minor modifications or specializations of the results in [10] are highly likely to yield the required results on Ψ_f.

Probabilistic questions.

We now motivate investigations into *randomly signed partition numbers,* which replace the function f in (2.1) with a random function from $\mathbb{N}$ to $\{0, \pm 1\}$. We first follow Harper's [16] succinct description of a Rademacher random multiplicative function (rmf.). A Rademacher rmf. is built by letting $(f(p))_{p \in \mathbb{P}}$ be independent Rademacher random variables, i.e., taking values ± 1 each with probability $1/2$, and setting $f(n) := \prod_{p|n} f(p)$ for all squarefree n, and $f(n) = 0$ when n is not squarefree. Rademacher rmfs. were introduced by Wintner [24] to model the Möbius μ function, and are thus natural candidates for generalizing our results on $(\mathfrak{p}(n, \mu))_{\mathbb{N}}$ to a probabilistic setting.

Next, we recall that a set $A \subset \mathbb{N}$ has *density* δ_A if the ratio $|A \cap \{1, 2, \ldots, N\}|/N$ tends to δ_A as $N \to \infty$. A remarkable theorem of Erdős [11] then states that: If $A \subset \mathbb{N}$ and $\gcd(A) = 1$, then A has density $\delta_A > 0$ if and only if

$$\log \mathfrak{p}(n, \mathbf{1}_A) \sim \kappa \sqrt{\delta_A n}. \tag{3.4}$$

From a probabilistic point of view, the density δ_A of a set $A \subset \mathbb{N}$ may be thought of as the probability that a random $n \in \mathbb{N}$ is an element of A. Considering the above discussions together with the results of the previous section, we consider the following problem.

Problem 3.4. Let $f(n)$ be a Rademacher rmf.. Determine Δ_f such that

$$\limsup_{n \to \infty} \left| \frac{\operatorname{logsc} \mathfrak{p}(n, f)}{\log \mathfrak{p}(n, 1)} \right| \leq \Delta_f.$$

Toward resolving Problem 3.4 in a manner like that of Problem 3.2, a number of usable results already exist. For instance, results on Dirichlet series $\sum f(n)n^{-s}$ associated to Rademacher rmfs. are found in [24], and Lemma 3.3 above already provides certain bounds on functions $\Psi_f(z)$, regardless of the random nature of $f(n)$.

For the case of the major arcs of the Hardy-Littlewood method as described above, an adaptation of the results of [10] is likely to provide the required bounds for $\Psi_f(z)$ by leveraging results on the summatory functions $\sum_{n \leq x} f(n)$ of Rademacher rmfs.. In particular, a number of results regarding these sums have been recently established, notable among which is Harper's result [15] that $\mathbb{E}|\sum_{n \leq x} f(n)| \asymp \sqrt{x}/(\log\log x)^{1/4}$. This and other ongoing work in the community on similar bounds are likely to provide additional tools toward results concerning $\Psi_f(z)$ if the results of [10, 15] require additional specialization.

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