MA 642
Homework 1

Problem 2.3

Let G be the Green's function for a bdd domain Ω. Prove

a) $G(x,y) = G(y,x)$ For all $x, y \in \Omega$, $x \neq y$

b) $G(x,y) < 0$

c) $\int_\Omega G(x,y) f(y)\,dy \to 0$ as $x \to \partial\Omega$, if f is bdd and $f \in L^1(\Omega)$.

a) (I learnt this proof in a previous class)

Let $y \in \Omega$, we write $G(x,y)$ as

$$G(x,y) = \Gamma(y-x) - h_y(x)$$

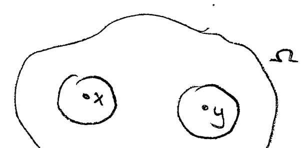
where $h_y(x)$ satisfies

$$\Delta_x h_y(x) = 0, \quad x \in \Omega$$
$$h_y(x) = \Gamma(y-x), \quad x \in \partial\Omega$$

Fix $x, y \in \Omega$, $x \neq y$. Let $z \in \partial\Omega$, then

$$G(x,z) = \Gamma(z-x) - h_z(x) = \Gamma(z-x) - \Gamma(z-x) = 0$$
$$G(y,z) = \Gamma(z-y) - h_z(y) = 0$$
$$\Delta_z G(x,\cdot) = \Delta_z G(y,\cdot) = 0 \quad \text{for } x \neq z,\ y \neq z.$$

and $G(x,\cdot) \in C^\infty(\Omega \setminus V_x)$, $G(y,\cdot) \in C^\infty(\Omega \setminus V_y)$ where V_x, V_y are small ngbhs of x and y, respectively.



Define u and v by

$$u(z) = G(x,z)$$
$$v(z) = G(y,z)$$

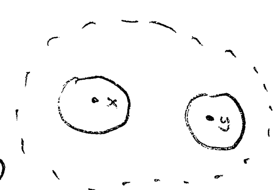
Let $\varepsilon > 0$ such that $u \in C^\infty(\Omega \setminus B_\varepsilon(x))$, $v \in C^\infty(\Omega \setminus B_\varepsilon(y))$ and $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$. By the Green's first identity we have

$$\int_{\Omega \setminus (B_\varepsilon(x) \cap B_\varepsilon(y))} \Delta u \cdot v\,dz = -\int_{\Omega \setminus (B_\varepsilon(x) \cap B_\varepsilon(y))} \nabla u \nabla v\,dz + \int_{\partial(\Omega \setminus (B_\varepsilon(x) \cap B_\varepsilon(y)))} \frac{\partial u}{\partial \nu} v\,d\sigma(z)$$

$$= -\left(\int_{\partial(\Omega \setminus (B_\varepsilon(x) \cap B_\varepsilon(y)))} u \frac{\partial v}{\partial \nu}\,d\sigma(z) - \int_{\Omega \setminus (B_\varepsilon(x) \cap B_\varepsilon(y))} u \Delta v\,dz \right) + \int_{\partial(\Omega \setminus (B_\varepsilon(x) \cap B_\varepsilon(y)))} \frac{\partial u}{\partial \nu} v\,d\sigma(z)$$ ← integration by parts

Since $\Delta u = \Delta v$ in $\Omega \setminus (B_\varepsilon(x) \cap B_\varepsilon(y))$ we have

$$0 = -\int_{\partial(\Omega \setminus ((B_\varepsilon(x)) \cap (B_\varepsilon(y))))} u \frac{\partial v}{\partial \nu}\,d\sigma(z) + \int_{\partial(\Omega \setminus (B_\varepsilon(x) \cap B_\varepsilon(y)))} \frac{\partial u}{\partial \nu} v\,d\sigma(z)$$



$\Omega \setminus (B_\varepsilon(x) \cap B_\varepsilon(y))$

$$\Leftrightarrow \int_{\partial\Omega} u \frac{\partial v}{\partial \nu}\,d\sigma(z) - \int_{\partial B_\varepsilon(x)} u \frac{\partial v}{\partial \nu}\,d\sigma(z) - \int_{\partial B_\varepsilon(y)} u \frac{\partial v}{\partial \nu}\,d\sigma(z) = \left(\int_{\partial\Omega} - \int_{\partial B_\varepsilon(x)} - \int_{\partial B_\varepsilon(y)} \right) v \frac{\partial u}{\partial \nu}\,d\sigma(z)$$

since $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$ and $B_\varepsilon(x), B_\varepsilon(y) \subset\subset \Omega$.

$$\Leftrightarrow \int_{\partial B_\varepsilon(x)} u \frac{\partial v}{\partial \nu}\,d\sigma(z) + \int_{\partial B_\varepsilon(y)} u \frac{\partial v}{\partial \nu}\,d\sigma(z) = \int_{\partial B_\varepsilon(x)} \frac{\partial u}{\partial \nu} v\,d\sigma(z) + \int_{\partial B_\varepsilon(y)} \frac{\partial u}{\partial \nu} v\,d\sigma(z)$$ since $u \equiv v \equiv 0$ on $\partial\Omega$

$$\Leftrightarrow \int_{\partial B_\varepsilon(x)} \left(\frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} \right) d\sigma(z) = \int_{\partial B_\varepsilon(y)} \left(u \frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu} v \right) d\sigma(z) \quad \forall \varepsilon > 0 \quad (*)$$

✓

Now, we have the following

u smooth near $y \Rightarrow \frac{\partial u}{\partial \nu}$ bounded near $\partial B_\varepsilon(y)$

v smooth near $x \Rightarrow \frac{\partial v}{\partial \nu}$ bounded near $\partial B_\varepsilon(x)$

also, h_x and h_y are smooth $\Rightarrow$ bounded.

<u>Notation</u>

We write $A \lesssim B$ if $\exists C>0$ such that $A \le CB$

$A \approx B$ if $\exists C_1, C_2 > 0$ such that $A \le C_1 B$ and $B \le C_2 A$

Then

$$\left|\int_{\partial B_\varepsilon(x)} u \frac{\partial v}{\partial \nu} d\sigma(z)\right| \lesssim \left|\int_{\partial B_\varepsilon(x)} u \, d\sigma(z)\right| \lesssim \sup_{\partial B_\varepsilon(x)} |u| \int_{\partial B_\varepsilon(x)} d\sigma(z) \lesssim \varepsilon^{n-1} \sup_{\partial B_\varepsilon(x)} |u| \approx \begin{cases} \varepsilon^{n-1} \cdot \varepsilon^{-n+2} & \text{if } n \ge 3 \\ \varepsilon^{n-1} \ln|\varepsilon| & \text{if } n=2 \end{cases}$$

$$\approx \begin{cases} \varepsilon & \text{if } n \ge 3 \\ \varepsilon \ln|\varepsilon| & \text{if } n=2 \end{cases} \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

and

$$\left|\int_{\partial B_\varepsilon(y)} \frac{\partial u}{\partial \nu} v \, d\sigma(z)\right| \lesssim \varepsilon^{n-1} \sup_{\partial B_\varepsilon(y)} |v| \approx \begin{cases} \varepsilon^{n-1} \varepsilon^{-n+2} & \text{if } n \ge 3 \\ \varepsilon^{n-1} \ln|\varepsilon| & \text{if } n=2 \end{cases} \approx \begin{cases} \varepsilon & \text{if } n \ge 3 \\ \varepsilon \ln|\varepsilon| & \text{if } n=2 \end{cases} \longrightarrow 0 \quad \text{as } \varepsilon \to 0$$

Also

$$\left|\int_{\partial B_\varepsilon(x)} \left(\frac{\partial}{\partial \nu} h_z(x)\right) v(z) \, d\sigma(z)\right| \lesssim \left|\int_{\partial B_\varepsilon(x)} d\sigma(z)\right| \approx \varepsilon^{n-1} \longrightarrow 0 \quad \text{as } \varepsilon \to 0 \quad \leftarrow \text{both } v, \frac{\partial h_z(x)}{\partial \nu} \text{ are bdd}$$

$$\left|\int_{\partial B_\varepsilon(y)} u(z) \left(\frac{\partial}{\partial \nu} h_z(y)\right) d\sigma(z)\right| \lesssim \left|\int_{\partial B_\varepsilon(y)} d\sigma(z)\right| \approx \varepsilon^{n-1} \longrightarrow 0 \quad \text{as } \varepsilon \to 0 \quad \leftarrow \text{both } u, \frac{\partial h_z(y)}{\partial \nu} \text{ bdd}$$

Finally

$$\int_{\partial B_\varepsilon(x)} \left(\frac{\partial}{\partial \nu} \Gamma(z-x)\right) v(z) \, d\sigma(z) = \begin{cases} \frac{1}{n\omega_n} \int_{\partial B_\varepsilon(x)} \frac{v(z)}{|z-x|^{n-1}} d\sigma(z) & n \ge 3 \\ \frac{1}{2\pi} \int_{\partial B_\varepsilon(x)} \frac{v(z)}{|z-n|} d\sigma(z) & n=2 \end{cases} = \frac{1}{|\partial B_\varepsilon(x)|} \int_{\partial B_\varepsilon(x)} v(z) \, d\sigma(z) \longrightarrow \underset{-v(x)}{\cancel{v(x)}} \quad \text{as } \varepsilon \to 0$$

and

$$\int_{\partial B_\varepsilon(y)} u(z) \left(\frac{\partial}{\partial \nu} \Gamma(z-y)\right) d\sigma(z) = \frac{1}{|\partial B_\varepsilon(y)|} \int_{\partial B_\varepsilon(y)} u(z) \, d\sigma(z) \longrightarrow u(y) \quad \text{as } \varepsilon \to 0$$

outward normal should be $x-z$

Therefore, by the limits above

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(x)} \left(\frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu}\right) d\sigma = v(x)$$

and

$$\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(y)} \left(u \frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu} v\right) d\sigma = \underset{-u(y)}{\underline{u(y)}}$$

$\Rightarrow$ by the equality (*) we conclude $u(y) = v(x)$.

Hence $G(x,y) = G(y,x)$.

b) Fix $y \in \Omega$. Define g on $\Omega \setminus \{y\}$ by

$$g(x) = G(x,y)$$

Since $\Gamma(x,y) \in C^2(\Omega \setminus \{y\})$ and $h_y \in C^2(\Omega) \cap C^1(\overline{\Omega})$ we have that if $\Omega_R = \Omega \setminus B_R(y)$ where $B_R(y) \subset\subset \Omega$ then

$$g \in C^2(\Omega_R) \cap C^0(\overline{\Omega}_R) \quad \text{for any } R \text{ such } B_R(y) \subset\subset \Omega.$$

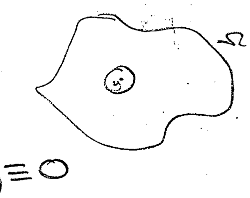
$h \in C^2(\Omega) \cap C^1(\overline{\Omega}) \Rightarrow h$ is bounded for any y, and by def. of Γ we have $\Gamma(|y-x|) \to -\infty$ as $x \to y$. Then, $g(x) \to -\infty$ as $x \to y$.

$\Rightarrow \exists R > 0$ with $\overline{B_R(y)} \subset\subset \Omega$ such that

$$g(x) < 0 \quad \text{for all } x: \ |x-y| \leq R$$

Now, $g \in C^2(\Omega_R) \cap C^0(\overline{\Omega}_R)$ and $\Delta g = 0$ on Ω_R for such $R \Rightarrow g$ satisfies the weak max/min principle strong minimum principle, so it satisfies

$$\Rightarrow \inf_{\partial(\Omega_R)} g \leq g(x) \leq \sup_{\partial \Omega_R} g, \quad x \in \Omega_R$$



Note that $\partial \Omega_R = \partial \Omega \cup \partial B_R(y) = \partial \Omega \cup \{|x-y| = R\}$ $\Rightarrow g(x) \leq 0$ on Ω_R

$\Rightarrow$ since $G = 0$ on $\partial \Omega$ $\Rightarrow g = 0$ on $\partial \Omega$ $\Rightarrow g \equiv 0$

since $G(x,y) \to -\infty$ as $x \to y \Rightarrow g$ is not constant

by the strong max/min principle and since ~~$g \equiv 0$~~ we have $g(x) < 0$ on $|x-y| = R$

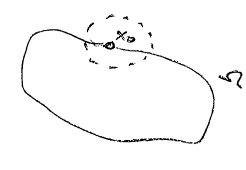
$\Rightarrow g(x) < 0$ on Ω_R

Taking $R \to 0$ we have $g(x) < 0$ for any $x \neq y \Rightarrow G(x,y) < 0$ if $x \neq y$.

Since y is arbritary we conclude $G(x,y) < 0$ for $x \neq y$.

c) Let $f \in L^1(\Omega)$ bdd.

We know that $G(x,y) \in C^2(\Omega \setminus \{y\}) \cap C(\overline{\Omega} \setminus \{y\})$, $\Delta_x G(x,y) = 0 \ \forall x \neq y$ and $G(x,y) = 0$ if $x \in \partial \Omega$, $x \neq y$.



Fix $x_0 \in \partial \Omega$, consider $B_\rho(x_0)$ with $\rho > 0$. Then

$$\int_\Omega G(x,y) f(y)\,dy = \int_{\Omega \setminus B_\rho(x_0)} G(x,y) f(y)\,dy + \int_{\Omega \cap B_\rho(x_0)} G(x,y) f(y)\,dy$$

Let $M > 0$ be such that $|f(x)| \leq M \ \forall x \in \Omega$, then

$$\left| \int_{\Omega \cap B_\rho(x_0)} G(x,y) f(y)\,dy \right| = \left| \int_{\Omega \cap B_\rho(x_0)} (\Gamma(y-x) + h_y(x)) f(y)\,dy \right| = \begin{cases} \left| \int_{\Omega \cap B_\rho(x_0)} \left(\frac{1}{n(n-2)\omega_n} \cdot \frac{1}{|x-y|^{n-2}} + h_y(x) \right) f(y)\,dy \right|, & n \geq 3 \\ \left| \int_{\Omega \cap B_\rho(x_0)} \left(\frac{1}{2\pi} \ln|x-y| + h_y(x) \right) f(y)\,dy \right|, & n = 2 \end{cases}$$

$$\lesssim \begin{cases} M \left| \int_{\Omega \cap B_\rho(x_0)} \frac{1}{|x-y|^{n-2}}\,dy \right| + \left| \int_{\Omega \cap B_\rho(x_0)} dy \right| & \text{if } n \geq 3 \\ M \ln|\rho| \left(\left| \int_{\Omega \cap B_\rho(x_0)} dy \right| + \left| \int_{\Omega \cap B_\rho(x_0)} dy \right| \right) & \text{if } n = 2 \end{cases} \quad \text{since } h_y \in C^2(\Omega) \cap C^1(\overline{\Omega}) \text{ is bdd}$$

$$\approx \begin{cases} \frac{1}{\rho^{n-2}} \rho^n + \rho^n & \text{if } n \geq 3 \\ \rho^n \ln|\rho| + \rho^n & \text{if } n = 2 \end{cases} \longrightarrow 0 \quad \text{as } \rho \to 0$$

Now, if we fix $\rho > 0$ we have that

$$\lim_{x \to x_0} G(x,y) = G(x_0, y) = 0, \quad y \in \Omega \setminus B_\rho(x_0) \text{ since } G \text{ is continuous.}$$

Let $\varepsilon>0$, then $\exists\ \rho_1>0$ such that

$$\left|\int_{\Omega\cap B_\rho(x_0)} G(x,y)f(y)dy\right| < \varepsilon/2 \quad \text{For all } 0<\rho\le\rho_1$$

$\Rightarrow$ FIX ρ, $0<\rho\le\rho_1$ $\Rightarrow$ since $C^2(\Omega\setminus\{y\})\cap C(\bar\Omega\setminus\{y\})$ we have

$$\lim_{\substack{x\to x_0\\ x\in\Omega}} \int_{\Omega\cap B_\rho(x_0)} G(x,y)f(y)dy = 0$$

by the unif. convergence of G in x near x_0 and in y away from x_0

$\Rightarrow \exists\ \rho_2>0$ such that

$$\left|\int_{\Omega\setminus B_\rho(x_0)} G(x,y)f(y)dy\right| < \varepsilon/2 \quad \text{with } |x-x_0|<\rho_2$$

$$\Rightarrow \int_\Omega G(x,y)f(y)dy \to 0 \text{ as } x\to\partial\Omega$$

Problem 2.4 (Schwarz reflection principle)

Let Ω^+ be a subdomain of the half space $x_n>0$ having as part of its boundary an open section T of the hyperplane $x_n=0$. Suppose that u is harmonic in Ω^+, $u\in C(\Omega^+\cup T)$, and that $u=0$ on T. Show that the function U defined by

$$U(x_1,\dots,x_n)=\begin{cases} u(x_1,\dots,x_n), & x_n\ge 0\\ -u(x_1,\dots,x_{n-1},-x_n), & x_n<0\end{cases}$$

is harmonic in the domain $\Omega^+\cup T\cup\Omega^-$, where Ω^- is the reflection of Ω^+ in $x_n=0$.

To prove: U satisfies the mean value property on $\Omega^+\cup T\cup\Omega^-$.

Let $B=B_R(y)\subset\subset(\Omega^+\cup T\cup\Omega)$ $\Rightarrow$ $y=(y_1,\dots,y_n)\in\Omega^+\cup T\cup\Omega^-$

Case 1: If $B_R(y)\subset\subset\Omega^+$, then $\partial B\subset\Omega^+$ and $y_n>0$

$$\frac{1}{n\omega_n R^{n-1}}\int_{\partial B} U ds = \frac{1}{n\omega_n R^{n-1}}\int_{\partial B} u\, ds \quad \text{since } U\equiv u \text{ in } \Omega^+$$

$$= u(y) \quad \text{since } u \text{ is harmonic in } \Omega^+$$

$$\Rightarrow \frac{1}{n\omega_n R^{n-1}}\int_{\partial B} U ds = u(y) = U(y), \quad \text{since } U(y)=u(y),\ y\in\Omega^+$$

$\Rightarrow U$ satisfies the mean value property in this case ✓

Case 2: If $B_R(y)\subset\subset\Omega^-$ $\Rightarrow \partial B\subset\Omega^-$ and $y_n<0$

We have

$$\frac{1}{n\omega_n R^{n-1}}\int_{\partial B} U ds = \frac{1}{n\omega_n R^{n-1}}\int_{\partial B} -u(x_1,\dots,-x_n)ds(x) \quad \leftarrow U(x_1,\dots,x_n) = -u(x_1,\dots,-x_n) \text{ in } \Omega^-$$

$$= \frac{-1}{n\omega_n R^{n-1}}\int_{\partial B} u(x_1,\dots,-x_n)ds(x)$$

$$= \frac{-1}{n\omega_n R^{n-1}}\int_{\partial B^-} u(x_1,\dots,x_n)(-1)ds(x)$$

where we made the change of variable $x_n\mapsto -x_n$; ∂B^- denotes the change of direction after this

$$= \frac{-1}{n\omega_n R^{n-1}}\int_{\partial B} u(x_1,\dots,x_n)ds(x)$$

$$= -u(y_*) \quad \text{where } y_* = (y_1,\dots,-y_n)\in\Omega^+$$

$$= -u(y_1,\dots,-y_n)$$

$$= U(y_1,\dots,y_n)$$

$\Rightarrow U$ satisfies the mean value property ✓

Then, U is harmonic in Ω^+ and in Ω^-. Then it is enough to prove that U satisfies the mean value for small balls for every $y \in T$.

Case 3

$B_R(y) \cap \Omega^+ \neq \emptyset$ and $B_R(y) \cap \Omega^- \neq \emptyset$ with $y \in T \Rightarrow y_n = 0$.

We have

$$\frac{1}{n\omega_n R^{n-1}} \int_{\partial B} U\,ds = \frac{1}{n\omega_n R^{n-1}} \int_{\substack{(x_1,\dots,x_n)\in\partial B \\ x_n>0}} U(x_1,\dots,x_n)\,ds + \frac{1}{n\omega_n R^{n-1}} \int_{\substack{(x_1,\dots,x_n)\in\partial B \\ x_n<0}} U(x_1,\dots,x_n)\,ds + \underbrace{\frac{1}{n\omega_n R^{n-1}} \int_{\substack{(x_1,\dots,x_n)\in\partial B \\ x_n=0}} U(x_1,\dots,x_n)\,ds}_{\substack{\| \\ 0 \\ \text{since } u=0 \\ \text{if } x_n=0}}$$

$$= \frac{1}{n\omega_n R^{n-1}} \int_{\substack{(x_1,\dots,x_n)\in\partial B \\ x_n>0}} u(x_1,\dots,x_n)\,ds + \frac{1}{n\omega_n R^{n-1}} \int_{\substack{(x_1,\dots,x_n)\in\partial B \\ x_n<0}} -u(x_1,\dots,-x_n)\,ds$$

$$= \frac{1}{n\omega_n R^{n-1}} \int_{\substack{(x_1,\dots,x_n)\in\partial B \\ x_n>0}} u(x_1,\dots,x_n)\,ds - \frac{1}{n\omega_n R^{n-1}} \int_{\substack{(x_1,\dots,-x_n)\in\partial B \\ x_n<0}} u(x_1,\dots,-x_n)\,ds$$

$= 0$ since both integrals are equal (the integral over the upper half ball cancels that over the lower half ball)

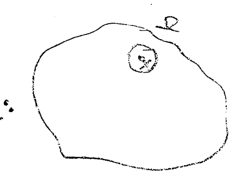
$= U(y)$ since $U(y) = u(y) = 0$ $\Rightarrow U$ satisfies the mean value property in $\Omega^+ \cup T \cup \Omega^-$ $\Rightarrow U$ is

Therefore, U satisfies the mean value property in $\Omega^+ \cup T \cup \Omega^-$ $\Rightarrow U$ is harmonic in $\Omega^+ \cup T \cup \Omega^-$.

Problem 2.7

Show that a $C^0(\Omega)$ function u is subharmonic in Ω iff it satisfies the mean value inequality locally; that is, for every $y \in \Omega$ there exists $\delta = \delta(y) > 0$ such that $u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(y)} u\,ds$ for all $R \leq \delta$.

$\Rightarrow$) Suppose u is subharmonic in Ω. Then $\forall B \subset\subset \Omega$, $\forall h$ harmonic in B with $u \leq h$ on ∂B we have $u \leq v$ in B.



Let $x \in \Omega$ and let $\delta(x) = \frac{1}{2}\,\mathrm{dist}(x, \partial\Omega)$ and $B_R = B_R(x) \subset \Omega$ for $0 < R \leq \delta(x)$. Let h be a harmonic in B_R such that $h = u$ on ∂B_R:

h is solution to $\begin{cases} \Delta h = 0 & \text{in } B_R \\ h = u & \text{in } \partial B_R \end{cases}$ for some $0 < R < \delta(x)$

then $u(x) \leq h(x) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R} h\,ds = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R} u\,ds$ (choose h for each R)

$\Rightarrow u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R} u\,ds$ for all $R \leq \delta(x)$

$\Rightarrow u$ satisfies the mean value inequality locally.

$\Leftarrow$) Suppose that u satisfies the mean value inequality locally. Let a ball $B_R(x) \subset\subset \Omega$ and h a harmonic function in Ω' with $B_R(x) \subset\subset \Omega' \subset \Omega$, such that $u \leq h$ on $\partial B_R(x)$.

Consider the function $w = u - h \Rightarrow w$ satisfies the mean value property locally (since h does) in $B_R(x)$; for any $y \in B_R(x)$ we have

$$w(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_{\delta(y)}(y)} w\,ds \quad \text{since } \partial B_{\delta(y)} \subset B_R(x)$$

Claim:

w satisfies the strong maximum principle

Proof:

Since $u, h \in C^0(\Omega') \Rightarrow w \in C^0(\Omega')$ (we have $w \in C(\overline{B_R})$) and satisfies the mean value prop. in B_R. Suppose $\exists x \in B_R$ such that $w(x) = \sup_{B_R} w$

Let $M = \sup_{B_R} w$ and define $\Omega_1 = \{y \in B_R \mid w(y) < M\}$ and $\Omega_2 = \{y \in B_R \mid w(y) = M\}$

Since w is continuous $\Rightarrow \Omega_1$ is open and Ω_2 is closed. Also, $x \in \Omega_2$ $\Rightarrow \Omega_2 \neq \phi$ and $B_R = \Omega_1 \cup \Omega_2$. Let $y \in \Omega_2$, then

$$M = u(y) \leq \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_{S(y)}(y)} w\, ds \leq M \quad \text{with } B_{S(y)}(y) \subset B_R(x)$$

$\Rightarrow w = M$ on the ball $B_{S(y)}(y) \subset B_R$ $\Rightarrow \Omega_2$ open

Since B_R is connected $\Rightarrow \Omega_2 = \phi$ or $\Omega_2 = B_R \Rightarrow \Omega_2 = B_R$ since $\Omega_2 \neq \phi \Rightarrow w$ is constant $\Rightarrow w$ satisfies the strong max. principle.

Since $w \leq 0$ on $\partial B_R \Rightarrow w \leq 0$ in $B_R \Rightarrow u \leq h$ in B_R.

Therefore, u is subharmonic.

Problem 2.8

An integrable function u in a domain Ω is called weakly harmonic (subharmonic, superharmonic) in Ω if

$$\int_\Omega u\, \Delta\varphi\, dx = (\geq, \leq)\, 0 \quad \forall\, \varphi \geq 0,\ \varphi \in C_0^2(\Omega).$$

Show that a $C^0(\Omega)$ weakly harmonic (subharmonic, superharmonic) function is harmonic (subharmonic, superharmonic).

Let ϕ be a radial smooth function. Assume that ϕ has support contained in $\{x : |x| \leq 1\}$, $\phi \geq 0$ and that ϕ has integral 1.

Denote ϕ_r by $\phi_r(x) = \frac{1}{r^n}\phi\left(\frac{|x|}{r}\right)$ and define for all $f \in L^1(\Omega)$ the function f_r by $f_r(x) = \phi_r * f(x)$

Then, let $\varphi \in C_0^2(\Omega) \Rightarrow \varphi_r \in C_0^\infty(\Omega)$. In particular, $\Delta\varphi_r \in C_0^2(\Omega)$, then by assumption $\int_\Omega u(y)\, \Delta\varphi_r(y)\, dy = (\geq, \leq)\, 0$

Note that

$$\int_\Omega u(y)\, \Delta\varphi_r(y)\, dy = \int_\Omega u(y)\, \Delta_y\left(\frac{1}{r^n}\int_\Omega \phi_r\left(\frac{|y-x|}{r}\right)\varphi(x)\, dx\right) dy = \int_\Omega u(y)\, \Delta_y\left(\frac{1}{r^n}\int_\Omega \phi_r\left(\frac{|x|}{r}\right)\varphi(y-x)\, dx\right) dy$$

$$= \int_\Omega u(y)\left(\frac{1}{r^n}\int_\Omega \phi_r\left(\frac{|x|}{r}\right)\Delta_y\, \varphi(y-x)\, dx\right) dy = \int_\Omega u(y)\left(\frac{1}{r^n}\int_\Omega \phi_r\left(\frac{|y-x|}{r}\right)\Delta\varphi(x)\, dx\right) dy$$

$$= \int_\Omega\int_\Omega \frac{1}{r^n}\phi_r\left(\frac{|y-x|}{r}\right) u(y)\, \Delta\varphi(x)\, dx\, dy = \int_\Omega\left(\frac{1}{r^n}\int_\Omega \phi\left(\frac{|y-x|}{r}\right) u(y)\, dy\right)\Delta\varphi(x)\, dx$$

$$= \int_\Omega u_r(y)\, \Delta\varphi(y)\, dy$$

$\Rightarrow \int_\Omega u_r(y)\, \Delta\varphi(y)\, dy = (\geq, \leq)\, 0$ for any $\varphi \in C_0^2(\Omega)$

Also, by smoothness of ϕ we have $u_r \in C_0^\infty(\Omega)$, then

$$\int_\Omega u_r(y)\, \Delta\varphi(y)\, dy = \int_\Omega \Delta u_r(y)\, \varphi(y)\, dy$$

$\Rightarrow \int_\Omega \Delta u_r(y)\varphi(y)dy = (\geq, \leq) 0 \quad \forall \varphi \in C_0^2(\Omega)$

Thus, $\Delta u_r = 0$ for any $r>0$.

Now, we have that $\|u_r\|_{L^1} \leq \|\phi_r\|_{L^1}\|u\|_{L^1} \leq \|u\|_{L^1}$

$\Rightarrow$ the sequence $\{u_{1/n}\}_{n\geq 1}$ is a bdd sequence of harmonic fnts $\Rightarrow \exists \{u_{n_k}\} \subset \{u_{1/n}\}$ that converges uniformly on $\Omega' \subset\subset \Omega$, say $u_{n_k} \to v \Rightarrow \Delta v = 0$ in $\Omega' \subset \Omega$

We also have, that by construction and def. of ϕ_r

$$u_r \longrightarrow u \text{ in } L^1(\Omega) \text{ as } r \to 0$$

$\Rightarrow u = v$ in Ω' $\Rightarrow \Delta u = (\geq, \leq)\, 0$

Since $u \in C_0^\infty(\Omega)$ we have $\quad \forall \varphi \in C_0^2(\Omega) \Rightarrow \Delta u = (\geq, \leq) 0$ in Ω

$$0 = (\geq, \leq) \int_\Omega u\,\Delta\varphi = \int \Delta u \cdot \varphi,$$

$\Rightarrow u$ is harmonic (subharmonic, superharmonic).

Problem 2.4

Show that for $C^2(\Omega)$ functions u, the conditions

i) $\Delta u \geq 0$ in Ω

ii) u is subharmonic in Ω

iii) u is weakly subharmonic in Ω, are equivalent.

i)$\Rightarrow$ii) Let $u \in C^2(\Omega)$ with $\Delta u \geq 0$ in Ω. Let $y \in \Omega$ and $R>0$ such that $B_R(y) \subset\subset \Omega$.
Let $\rho \in (0,R)$, by the divergence theorem applied to $B_\rho = B_\rho(y)$

$$\int_{\partial B_\rho} \frac{\partial u}{\partial \nu} ds = \int_{B_\rho} \Delta u\, dx \geq 0$$

also where $r = |x-y|$, $\omega = \frac{x-y}{|x-y|}$

$$\int_{\partial B_\rho} \frac{\partial u}{\partial \nu} ds = \int_{\partial B_\rho} \frac{\partial u}{\partial r}(y+\rho\omega)d\omega$$
$$= \rho^{n-1}\int_{|\omega|=1} \frac{\partial u}{\partial r}(y+\rho\omega)d\omega = \rho^{n-1}\frac{\partial}{\partial\rho}\int_{|\omega|=1} u(y+\rho\omega)d\omega = \rho^{n-1}\frac{\partial}{\partial\rho}\left[\rho^{1-n}\int_{\partial B_\rho} u\,ds\right]$$

$\Rightarrow \frac{1}{\rho^{n-1}}\int_{\partial B_\rho}\frac{\partial u}{\partial \nu}ds = \frac{\partial}{\partial\rho}\left[\frac{1}{\rho^{n-1}}\int_{\partial B_\rho} u\,ds\right] \geq 0 \quad \Rightarrow \frac{1}{\rho^{n-1}}\int_{\partial B_\rho} u\,ds$ is increasing on ρ

$\Rightarrow$ For any $\rho \in (0,R)$ we have

$$\frac{1}{\rho^{n-1}}\int_{\partial B_\rho} u\,ds \leq \frac{1}{R^{n-1}}\int_{\partial B_R} u\,ds$$

and $\lim_{\rho \to R} \frac{1}{\rho^{n-1}}\int_{\partial B_\rho} u\,ds = n\omega_n u(y)$

$\Rightarrow u(y) \leq \frac{1}{n\omega_n R^{n-1}}\int_{\partial B_R} u\,ds \quad \Rightarrow u$ satisfies the mean value property, in particular it satisfies the mean value locally (and $u \in C^0(\Omega)$) $\Rightarrow$ by problem 2.7 we conclude that u is subharmonic in Ω.

iii)$\Rightarrow$ii). This follows by problem 2.8.

ii)$\Rightarrow$i) Let u subharmonic in Ω, $u \in C^2(\Omega) \Rightarrow u \in C^0(\Omega)$. By problem 2.7, u satisfies the mean value property locally.

Suppose $\exists\, x_0 \in \Omega$ such that $\Delta u(x_0) < 0$. Let $R>0$ be such that $B_R(x_0) \subset\subset \Omega$

Since $u \in C^2(\Omega) \Rightarrow \Delta u$ is continuous in Ω $\Rightarrow \exists\ R_1 > 0$ such that
$\Delta u(x) < 0\ \forall\ x \in B_\rho(x_0)$ with $0 < \rho < R_1$

Then if $B_\rho = B_\rho(x_0)$ we have as above for $0 < \rho < R_1$

$$0 > \frac{1}{\rho^{n-1}} \int_{B_\rho} \Delta u\, dx = \frac{\partial}{\partial \rho}\left(\frac{1}{\rho^{n-1}} \int_{\partial B_\rho} u\, ds\right) \Rightarrow \frac{1}{\rho^{n-1}} \int_{\partial B_\rho} u\, ds \text{ is strictly decreasing on } \rho$$

$$\Rightarrow \frac{1}{\rho^{n-1}} \int_{\partial B_\rho} u\, ds > \frac{1}{R^{n-1}} \int_{\partial B_R} u\, ds \quad \forall\ 0 < \rho < R^{n-1}$$

since $\lim_{\rho \to R} \frac{1}{\rho^{n-1}} \int_{\partial B_\rho} u\, ds = n\omega_n u(x_0)$ $\Rightarrow u(x_0) > \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R} u\, ds$ which contradicts that u satisfies the mean value property $\Rightarrow$ there is not $x_0 \in \Omega$ such that $\Delta u(x_0) < 0$. Thus, $\Delta u \geq 0$ in Ω

i) $\to$ iii) Suppose $u \in L^1(\Omega)$, $u \in C^2(\Omega)$ and $\Delta u \geq 0$ in Ω.
Let $\varphi \in C_0^2(\Omega)$ be such that $\varphi \geq 0$. Then by integration by parts

$$\int_\Omega (u \Delta \varphi - \Delta u \cdot \varphi)\, dx = \int_{\partial\Omega} \left(u \frac{\partial \varphi}{\partial \nu} - \frac{\partial u}{\partial \nu} \varphi\right) d\sigma$$

since φ has compact support in Ω $\Rightarrow \varphi \equiv 0$ on $\partial\Omega$, $\Rightarrow \int_{\partial\Omega} \left(u \frac{\partial \varphi}{\partial \nu} - \frac{\partial u}{\partial \nu} \varphi\right) d\sigma = 0$

$$\Rightarrow \int_\Omega u \Delta \varphi\, dx = \int_\Omega \Delta u \cdot \varphi\, dx$$

Since $\varphi \geq 0$ and $\Delta u \geq 0$ in Ω we have that $\int_\Omega \Delta u \cdot \varphi\, dx \geq 0$

Therefore, $\int_\Omega u \Delta \varphi\, dx \geq 0$

Thus, we have proven that
i) $\Rightarrow$ iii) $\Rightarrow$ ii) $\Rightarrow$ i)

$\Rightarrow$ the conditions are equivalent

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