§2.

11. Given a point on $\partial\Omega$, we can choose an orthogonal coordinate system for $\mathbb{R}^n$ s.t the point lies at $(0,\cdots,0)$ and the exterior normal at Ω of the point is $(0,0,\cdots,0,1)$

then since $\partial\Omega$ is C^2, $\exists f: R^{n-1} \to R^n$ f continously diff

$f(x_1,x_2,\cdots x_{n-1}) = x_n$ denote $(x_1,x_2,\cdots x_{n-1})$ by x'

then $f(x') = x_n$

Such that, $\exists\, \varepsilon>0$, $\Omega \cap B(0,\varepsilon) = \{x_n < f(x')\}$

Note that $f(0)=0$, $Df(0)=0$ by Taylor theorem, $|f(x')| \le M\|x'\|_2^2$ for some x' in the neighbourhood of 0. $= M(x_1^2+x_2^2+\cdots x_{n-1}^2)$

Let $y = \delta e_n$, $\delta>0$, where $e_n = (0,0,\cdots 0,1)$

Claim, we can have $B(y,\delta) \subset \mathbb{R}^n \setminus \bar{\Omega}$, $\overline{B(y,r)} \cap \bar{\Omega} = \{0\}$ for a suitable δ

Since, if $x \in \overline{B(y,\delta)}$, $x \neq 0$, then $|x-y|^2 = |x|^2 - 2x\cdot y + |y|^2$
$= |x|^2 - 2\delta x_n + \delta^2 \le \delta^2$

$\Rightarrow |x|^2 \le 2\delta x_n$

hence $f(x') \le M\|x'\|_2^2 \le M\|x\|_2^2 \le 2M\delta x_n < x_n$

if $\delta < \frac{1}{2M}$, then $x \in \mathbb{R}^n \setminus \Omega$

12. We can also write the exterior cone condition to be as follows

there exists $x_0 \in R^n \setminus \{\xi\}$, $0<\theta_0<\pi$, $r_0>0$

$$\Omega \cap B(\xi, r_0) \subset \{x : \text{angle}(\xi - x_0, x-\xi) < \theta_0\}$$

Suppose $n \ge 3$. let $w(x) = -r^\lambda f(\theta)$ $r = |x-\xi|$, $\theta = \text{angle}(\xi - x_0, x-\xi)$

$\theta \in [0,\theta_0]$, we try to seek $f(\theta)$ s.t $\Delta w \ge 0$ in $\Omega \setminus \{\xi\}$

Denoting by x' the projection of x on the line $D = \xi + \mathbb{R}(x_0 - \xi)$

$$\rho = r\sin\theta = \|x'-x\|, \quad \nu = r\cos\theta = x'-\xi$$

And by the formula for the Laplacian in cylindrical coordinates

$$\Delta w = \frac{\partial^2 w}{\partial \nu^2} + \frac{\partial^2 w}{\partial \rho^2} + \frac{n-2}{\rho}\frac{\partial w}{\partial \rho} \quad \text{in } \Omega \setminus D$$

passing to polar coordinates (r,θ) in the $\nu\rho$-plane,

and $\dfrac{\partial w}{\partial \rho} = \dfrac{\partial w}{\partial r}\dfrac{\rho}{r} + \dfrac{\partial w}{\partial \theta}\dfrac{\cos\theta}{r}$

$$\Delta w = \frac{\partial^2 w}{\partial r^2} + \frac{n-1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2 w}{\partial \theta^2} + (n-2)\cot\theta \cdot \frac{\partial w}{\partial \theta}\right)$$

$$= \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial w}{\partial r}\right) + \frac{1}{r^2 \sin\theta^{n-2}}\frac{\partial}{\partial \theta}\left(\sin\theta^{n-2}\frac{\partial w}{\partial \theta}\right)$$

$$= -r^{\lambda-2}\left(\lambda(\lambda+n-2)f(\theta) + \frac{1}{\sin\theta^{n-2}}\frac{d}{d\theta}\left(\sin\theta^{n-2}\frac{df}{d\theta}(\theta)\right)\right)$$

fix $\theta_0 < \theta_0' < \pi$ let

$$f(\theta) = \int_0^{\theta_0'} \left(\frac{t}{\sin t}\right)^{n-2} dt$$

$$\Rightarrow \Delta w = r^{\lambda-2}\left(\frac{(n-2)\theta^{n-3}}{\sin\theta^{n-2}} - \lambda(\lambda+n-2)f(\theta)\right) \text{ on } \Omega \setminus D$$

Since $f(\theta) \le f(0)$, and $\frac{(n-2)\theta^{n-3}}{\sin\theta^{n-2}} \geq \frac{n-2}{\theta_0}$.

choose $\lambda = \frac{n-2}{2}\left(\left(1+\frac{4}{\theta_0^2 f(0)^2}\right)^{1/2} - 1\right)$

then $\lambda > 0$. $\Delta w \geq 0$ on $\Omega \setminus D$ $(n \geq 3)$ ✓

Also do $n=2$.

14.

(a) By the harnack inequality, for $B(0,r)$

Here, let $m = \inf_{\mathbb{R}^n} u$, replacing u by $u-m$, we may assume $m=0$ and hence u non-negative

by Harnack inequality $\sup_{B(0,r)} \le 3^n \inf_{B(0,r)} u$

3^n is independent of r, let $r \to \infty \Rightarrow u \equiv 0$ ✓

hence, u is constant

(b) If $\Delta v \geq 0$ in $\mathbb{R}^2$, then by hadamard's thm. let $M(r) = \max \{v(x) : \|x\| = r\}$,

$$M(r) \le M(a)\frac{\log(b) - \log(r)}{\log(b) - \log(a)} + M(b)\cdot\frac{\log(r) - \log(a)}{\log(b) - \log(a)}$$

for $a < r < b$

Since $M(r)$ is bounded above, let $b \to \infty$, $a \to 0$

we have $M(r) \le M(a)$ for $r \geq a$

$M(r) \le M(b)$ for $r \le b$

$\Rightarrow M(r)$ is constant on $(0, \infty)$

by strong maximum principle, v is constant ✓

(c) Solve $\Delta u = \rho$. ρ is a smooth function and $\rho \geq 0$, $\int_{\mathbb{R}^n} \rho\,dx = 1$.

An analytic solution:

$$u(\xi) = \int_{\mathbb{R}^n} K(x,\xi)\rho(x)\,dx$$

What is K?

let $\rho = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}}$

$$|u(\xi)| = \frac{1}{(n-2)\omega_n(2\pi)^{n/2}} \int_{\mathbb{R}^n} |x-\xi|^{2-n} e^{-|x|^2/2}\,dx$$

Split into case where $|x-\xi|>1$ and $|x-\xi|\leq 1$

then $\int_{|x-\xi|\geq 1} |x-\xi|^{2-n} e^{-|x|^2/2}\,dx \leq \int_{\mathbb{R}^n} e^{-|x|^2/2}\,dx = (2\pi)^{n/2}$

$$\int_{|x-\xi|\leq 1} |x-\xi|^{2-n} e^{-|x|^2/2}\,dx \leq \int_{|x-\xi|\leq 1} |x-\xi|^{2-n}\,dx = \omega_n \int_0^1 r\,dr = \frac{\omega_n}{2}$$

$\Rightarrow$ u is bounded ✓

§3. 1. (a) Suppose $u \not\equiv$ constant. If there exists $x \in \Omega$ such that $u(x) > 0$, then $\exists\, x_0 \in \partial S_2$ s.t $\sup_{x\in\Omega} u = u(x_0)$. W.L.O.G assume x_0 is the origin and $\vec{\nu}$ is pointing to the negative x_n-axis. And B is the interior ball at $x_0 \in S_2$, Since u is not constant, we know that $u(x_0) > 0$

$\Rightarrow D_n u(x_0) > 0$

Since $u \in C^1(\Omega \cup S_2)$, then $u \in C^1(B \cup x_0)$. So $D_i u(x_0) = 0$ for $i = 1, 2, \cdots n-1$. $\Rightarrow$ $\beta(x_0)\cdot \vec{D}(u) = 0$ means $\beta_n = 0$, contradicts to the fact that β has non-zero component at each point $\Rightarrow u \leq 0$ in Ω ✓ Similarly, $-u \leq 0$ $\Rightarrow u = 0$

(b) Suppose $u \not\equiv$ constant. If $\exists\, x \in \Omega$ such that $u(x) > 0$ then $\exists\, x_0 \in \partial\Omega$, s.t $\sup_{x\in\Omega} u = u(x_0)$ W.L.O.G assume x_0 is the origin and $\vec{\nu}$ is pointing to the negative x_n-axis. And B is the interior ball at $x_0 \in S_2$. Since u is not constant and $u \in C^1(B \cup x_0)$ then $D_i u(x_0) = 0$ for $i = 1, 2, \cdots n-1$. $D_n u(x_0) > 0$

$\Rightarrow \alpha(x_0)u(x_0) + \beta \cdot Du = \alpha(x_0)u(x_0) + \beta_n(x_0) D_n u(x_0) = 0$

Since $u(x_0) > 0$, $D_n u(x_0) > 0$, $\Rightarrow \alpha(x_0)\beta_n(x_0) < 0$

but $\alpha(x_0)\beta_n(x_0) = \cancel{\alpha}\cdot(\beta\cdot\nu) > 0$
contradiction! ✓

hence $u \le 0$ in Ω. Similarly $-u \le 0$, $u \equiv 0$

2.(a) Assume that u assumes its maximum M at an interior point and let $\Omega^- = \Omega \cap \{u < M\}$. If Ω^- is not empty, then $\partial\Omega^- \cap \Omega$ is not empty. Let y be a point in Ω^- that is closer to $\partial\Omega^-$ than to $\partial\Omega$ and let B be the largest ball contained in Ω^- and centered at y. Let x_0 be a point on $\partial B \cap \partial\Omega^-$, then, by lemma 3.4, to the ball B, ∇u is nonzero at x_0, contradicting the assumption that u assumes its maximum there

(b) Consider $\frac{Lu}{-c} = \frac{f}{-c}$, by thm. 3.7. $\sup_\Omega |u| \le \sup_{\partial\Omega} |u| + \tilde{C} \sup_\Omega \frac{|f|}{|c|} \cdot \frac{1}{\hat{\lambda}}$

where $\hat{\lambda} = -\frac{\lambda}{c}$, $\hat{C} = e^{(\tilde{\beta}+1)d} - 1$

$$\tilde{\beta} = \sup \frac{|\tilde{b}|}{|\hat{\lambda}|} = \sup \frac{|b|/_{-c}}{|\lambda|/_{-c}} = \sup \frac{|b|}{|\lambda|} = \beta$$

$$\Rightarrow \tilde{C} = e^{(\beta+1)d} - 1$$

as $d \to 0$, $\hat{C} \to 0$, hence when d sufficiently small.

$\frac{\hat{C}}{\hat{\lambda}} < 1$.

$$\Rightarrow \sup_\Omega |u| \le \sup_{\partial\Omega} |u| + \frac{\hat{C}}{\hat{\lambda}} \sup \left|\frac{f}{c}\right| \le \sup_{\partial\Omega} |u| + \sup_\Omega \frac{|f|}{|c|}$$

3.8 If $u = u(r)$, then $D_i u = u' \frac{x_i}{r}$

$$D_{ij} u = u'' \frac{x_i x_j}{r^2} + u' \frac{\delta_{ij} r - x_i x_j \cancel{r}}{r^2}$$

$$L_n u = a^{ij} D_{ij} u = \sum_{i,j} \left(\delta^{ij} + g(r) \frac{x_i x_j}{r^2}\right) D_{ij} u$$

$$= \sum_{i,j} \left(\delta^{ij} + g(r) \frac{x_i x_j}{r^2}\right) \left(u'' \frac{x_i x_j}{r^2} + u' \left(\frac{\delta_{ij} r - \frac{x_i x_j}{r}}{r^2}\right)\right)$$

$$= u'' + \frac{n-1}{r} u' + g(r) \left(u'' + \frac{u'}{r} - \frac{u'}{r}\right) = 0$$

$$\Rightarrow (1 + g(r)) u'' + \frac{n-1}{r} u' = 0$$

$$\frac{u''}{u'} = \frac{1-n}{r(1+g)}$$

a) if $n = 2$
$$A(x) = \begin{bmatrix} 1 + g(r) \frac{x_1^2}{r^2} & g(r) \frac{x_1 x_2}{r^2} \\ g(r) \frac{x_1 x_2}{r^2} & 1 + g(r) \frac{x_2^2}{r^2} \end{bmatrix}$$

3.8

(a) If $u = u(r)$, $\quad D_i u = u' \frac{x_i}{r} \quad (\frac{x_i}{r})$

$$D_{ij} u = u'' \frac{x_i x_j}{r^2} + u' \left(\frac{\delta_{ij} r - \frac{x_i x_j}{r}}{r^2} \right)$$

$$L_n u = a^{ij} D_{ij} u = u'' + \frac{n-1}{r} u' + g(r)\, u'' \frac{\sum_{i,j=1}^{n} x_i^2 x_j^2}{r^4} + g(r) \frac{x_i x_j}{r^2} u' \left(\frac{\delta_{ij} - x_i x_j}{r^2} \right)$$

$$= u'' + \frac{n-1}{r} u' + g(r) \left(u'' + \frac{u'}{r} - \frac{u'}{r} \right) = 0$$

$$\Rightarrow \quad (1 + g(r))\, u'' + \frac{n-1}{r} w = 0$$

$$\frac{u''}{u'} = \frac{1-n}{r(1+g)}$$

If $n = 2$, $g(r) = -2/(2 + \log r)$

$$a_{ij} = \delta_{ij} + g(r) \frac{x_i x_j}{r^2} = \delta^{ij} - \frac{2 x_i x_j}{(2 + \log r) r^2} = \begin{cases} - \dfrac{2 x_i x_j}{(2 + \log \sqrt{x_i^2 + x_j^2})(x_i^2 + x_j^2)} & i \neq j \\ 1 - \dfrac{1}{2 + \log \sqrt{x_i^2}} & i = j \end{cases}$$

Obviously, a^{ij} is continous on $\mathbb{R}^2/\{0\}$ only need to prove a^{ij} also continuous at $\{0\}$.

Since $\lim_{\substack{x_i \to 0 \\ x_j \to 0}} \frac{x_i}{2+\log r} = 0$ and $\frac{|x_i|}{r^2} \le 1$, $\frac{|x_j|}{r^2} \le 1$.

We will have $\lim_{\substack{x_i \to 0 \\ x_j \to 0}} a^{ij} = \begin{cases} 0 & i \ne j \\ 1 & i = j. \end{cases}$

i.e. a^{ij} is continous on $\mathbb{R}^2$. Hence L_n is Uniformly elliptic in the bdd domain D.

(b) If $n > 2$, $g(r) = -[1+(n-1)\log r]^{-1}$.

For $0 \le r \le e^{-1}$, $g(r) \le -\frac{1}{n}$.

$$a^{ij}\xi_i\xi_j = |\xi|^2 + g(r)\frac{x_i x_j}{r^2}\xi_i\xi_j$$
$$\ge |\xi|^2 - \frac{x_i x_j}{n r^2}\xi_i\xi_j$$
$$\ge (1-\tfrac{1}{n})|\xi|^2.$$ uniformly elliptic.

Use the fact that the maximum eigenvalue of $\left(\frac{x_i x_j}{n r^2}\right)$ is $\frac{1}{n}$.

$r \ll 1 \Rightarrow g \sim 0$

$$\frac{u''}{u'} = \frac{1-n}{r(1+g)} \sim \frac{u''}{u'} \approx \frac{1-n}{r}$$
$$(\log u')' \approx \frac{1-n}{r}$$
$$\log u' \approx C(1-n)\log r$$
$$u' \approx C \cdot r^{1-n}.$$
$$u' = O(r^{2-n}) \quad \text{as } r \to 0.$$

(c). Consider $g(r) = -\frac{n}{(n+\log r)}$.