11/4/19 - Spring mass systems —

$\frac{d^2x}{dt^2} + \frac{c}{m}\frac{dx}{dt} + \frac{kx}{m} = \frac{F(t)}{m}$, where k is spring constant, and c is the damping constant. Here $c \geq 0$, $m > 0$, $k > 0$.

Case 1 $c = 0$, $F(t) = 0$, no damping and no external force

$$x(t) = c_1 \cos\left(\sqrt{\tfrac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\tfrac{k}{m}}\, t\right)$$

Case 2 $c = 0$, $F(t) = F_0 \cos(\omega t)$, $\omega_0 = \sqrt{\frac{k}{m}}$ (circular frequency)

(a) If $\omega \neq \omega_0$ $\quad x(t) = A_0 \cos(\omega_0 t - \phi) + \frac{F_0 \cos \omega t}{m(\omega_0^2 - \omega^2)}$

(b) (Resonance) $\quad x(t) = A_0 \cos(\omega_0 t - \phi) + \frac{F_0 t \sin(\omega_0 t)}{2m\omega_0}$

(subtract $\cos \omega_0 t$, take limit of (a)) or method of undetermined coefficients

Case 3 $c > 0$, $c^2 < 4km$ (underdamped), $F(t) = 0$

$x(t) = A_0 \exp(-ct/2m) \cos(\mu t - \phi)$, $\mu = \left(\frac{1}{2m}\right)\sqrt{4km - c^2}$

Case 4 $c > 0$, $c^2 > 4km$ (overdamped), $F(t) = 0$

$x(t) = \exp(-ct/2m)\left(c_1 e^{\mu t} + c_2 e^{-\mu t}\right)$, $\mu = \left(\frac{1}{2m}\right)\sqrt{c^2 - 4km}$

Do p. 495/6, 9, 14

Case 5 $\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m} y = \frac{F_0}{m}\cos(\omega t)$, $c > 0$

(a) Use method of complex trial solutions and Euler's formula to derive (6.5.24), p. 497/29. That is solve $d^2y/dt^2 + \frac{c}{m}\, dy/dt + \frac{k}{m} y = \frac{F_0}{m} e^{i\omega t}$, $y = Ae^{i\omega t}$. Obtain

$A = F_0 \frac{(k - m\omega^2) - ic\omega}{(k - m\omega^2)^2 + c^2\omega^2}$ Take $\mathrm{Re}(Ae^{i\omega t})$ to get (6.5.24), p. 493

Explain rewrite of (6.5.24) and $y_p = \frac{F_0}{H}\cos(\omega t - \eta)$ as in text.

(b) Combine with solution of homogeneous to get (6.5.26) in underdamped case. Explain transient and steady state solutions

$$y_h = A_0 e^{-ct/m} \cos(\mu t - \phi)$$

Do p. 497/ 30, 31, 14

4/9/20 — Mechanical vibrations — Follow notes of 11/4/19

Additional details for Case 5 of 11/4/19 — Forced oscillation with damping. Assume $\omega > 0$ and $c > 0$. Use complex trial solutions to solve

$$y'' + \frac{c}{m} y' + \frac{k}{m} y = \frac{F_0}{m} \cos\omega t = \frac{F_0}{m} \mathrm{Re}(e^{i\omega t})$$

Consider

$$y'' + \frac{c}{m} y' + \frac{k}{m} y = \frac{F_0}{m} e^{i\omega t}, \text{ trial solution}$$

$y = Ae^{i\omega t}$

Calculate $y' = Ai\omega e^{i\omega t}$, $y'' = -A\omega^2 e^{i\omega t}$

$$-A\omega^2 e^{i\omega t} + A\left(\frac{c}{m}\right) i\omega e^{i\omega t} + A\left(\frac{k}{m}\right) e^{i\omega t} = \frac{F_0}{m} e^{i\omega t}$$

$$A(k - m\omega^2 + i\omega c) = F_0$$

Thus

$$A e^{i\omega t} = \frac{F_0(k - m\omega^2 - ic\omega)}{(k - m\omega^2)^2 + c^2\omega^2} (\cos\omega t + i \sin\omega t)$$

and

$$y_p = \mathrm{Re}(Ae^{i\omega t}) = F_0((k - m\omega^2)\cos\omega t + c\omega \sin\omega t) / (k - m\omega^2)^2 + c^2\omega^2$$

Here y_p is a particular solution of $y'' + \frac{c}{m} y' + \frac{k}{m} y = F_0 \cos\omega t / m$.

It is useful to have an alternative expression for y_p. Let $\omega_0 = \left(\frac{k}{m}\right)^{1/2}$, so $k = m\omega_0^2$, giving

$$y_p = \frac{F_0}{(m\omega_0^2 - m\omega^2)^2 + c^2\omega^2} \left((m\omega_0^2 - m\omega^2)\cos\omega t + c\omega \sin\omega t\right)$$

$$= \frac{F_0}{H} \left(\cos\omega t \frac{m(\omega_0^2 - \omega^2)}{H} + \sin\omega t \frac{c\omega}{H}\right)$$

where $H = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}$. If the angle η is defined by

$$\cos\eta = \frac{m(\omega_0^2 - \omega^2)}{H} \quad \text{and} \quad \sin\eta = \frac{c\omega}{H}$$

then $y_p = \frac{F_0}{H}(\cos\omega t \cos\eta + \sin\omega t \sin\eta) = \frac{F_0}{H}\cos(\omega t - \eta)$

In the case of underdamping, the general solution is $y = A_0 e^{-ct/m} \cos(\mu t - \phi) + \frac{F_0}{H}(\cos(\omega t - \eta)) = y_h + y_p$, where

$$y_h = A_0 e^{-ct/m} \cos(\mu t - \phi), \text{ the transient solution}$$

and $y_p = \frac{F_0}{H} \cos(\omega t - \eta)$, the steady state solution

Problem 1 (a) Show that the time between successive maxima (or minima) of the transient solution is $T = \frac{2\pi}{\mu}$.

Calculate $y_h' = A\left(\frac{-c}{2m}\cos(\mu t - \phi) - \mu \sin(\mu t - \phi)\right) e^{\frac{-ct}{2m}}$, so $y_h' = 0$ exactly when $\tan(\mu t - \phi) = -c/2m\mu$. Note that the tangent function has period π, so the critical points of y_h are separated by π/μ. The graph of y_h indicates that maxima and minima alternate, so successive maxima and minima are separated by $2\pi/\mu$.

(b) Show that if $\frac{c^2}{4km} \ll 1$, then $T \approx 2\pi\sqrt{\frac{m}{k}}$.

Recall that $\mu = \sqrt{4km - c^2}/2m$, so $T = 4\pi m/\sqrt{4km - c^2}$ $= 2\pi\sqrt{\frac{m}{k}}/\sqrt{1 - c^2/4km} \approx 2\pi\sqrt{\frac{m}{k}}$. This agrees with the undamped case.

$4\pi m$

Problem 2 Assuming that $c^2/2m^2\omega_0^2 < 1$, show that the amplitude of the steady state solution is maximized when $\omega = [\omega_0^2 - c^2/2m^2]^{1/2}$.

First note that the maximum amplitude of y_p is attained when H is minimal. Let $x = \omega^2$, so that $H^2 = m^2(\omega_0^2 - x)^2 + c^2 x$. Thus H is minimal when $f(x) = m^2x^2 - 2m^2\omega_0^2 x + c^2 x + m^2\omega_0^4$ is minimal, restricted to the half line $x \geq 0$.

Calculate $f'(x) = 2m^2 x - 2m^2\omega_0^2 + c^2$, which vanishes when $x = \omega_0^2 - c^2/2m^2$. The hypothesis $c^2/2m^2\omega_0^2 < 1$ guarantees that $x > 0$. Since $x = \omega^2$ and the graph of $f(x)$ is a parabola, the steady state solution assumes its maximum at the required value.

Problem 3 Consider the damped spring mass system with $m = 1$, $k = 5$, $c = 2$ and $F(t) = 8\cos\omega t$.

(a) Determine the transient part of the solution. In general $y_h = A_0 e^{-ct/2m}\cos(\mu t - \phi)$, where $\mu = \sqrt{4km - c^2}/2m$. So in this specific case $y_h = A_0 e^{-t}\cos(2t - \phi)$

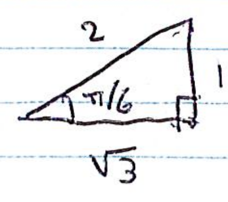
(b) Determine the steady state part of the solution. In general $y_p = \frac{F_0}{H}\cos(\omega t - \eta)$. Recall that $H^2 = c^2\omega^2 + m^2(\omega_0^2 - \omega^2)^2$, so that $H^2 = m^2\left(\frac{k}{m} - \omega^2\right)^2 + c^2\omega^2$. In this special case $H^2 = (\omega^2 - 5)^2 + 4\omega^2 = \omega^4 - 6\omega^2 + 25$. So $y_p = 8(\omega^4 - 6\omega^2 + 25)^{-1/2}\cos(\omega t - \eta)$. Alternatively, using the general formula $y_p = (F_0/H^2)((k - m\omega^2)\cos\omega t + c\omega\sin\omega t)$, and substituting the given numerical values we obtain $y_p = 8(\omega^4 - 6\omega^2 + 25)^{-1}((5 - \omega^2)\cos\omega t + 2\omega\sin\omega t)$

(c) Determine the value of ω that maximizes the amplitude of y_p and express y_p in the form $y_p = A_0\cos(\omega t - \eta)$.

By Problem 2, $[\omega_0^2 - c^2/(2m^2)]^{1/2}$ is the required value of ω, provided $c^2/(2m^2\omega_0^2) < 1$. Here $\omega_0 = \sqrt{k/m}$. For the given values $c^2/(2m^2\omega_0^2) = 2/5$ and the value of ω is $\sqrt{3}$, so $y_p = 2\cos(\sqrt{3}t - \eta)$, using (b). Moreover, using $H^2 = \omega^4 - 6\omega^2 + 25$ and the general formulas for $\cos\eta$ and $\sin\eta$, we get $\cos\eta = \frac{1}{2}$ and $\sin\eta = \frac{\sqrt{3}}{2}$. So $\omega = \pi/3$, and

$$y_p = 2\cos\left(\sqrt{3}t - \frac{\pi}{3}\right)$$

General formulas cited above are $\cos\eta = \frac{m(\omega_0^2 - \omega^2)}{H}$ and $\sin\eta = \frac{c\omega}{H}$. Here $\omega_0 = \sqrt{5}$, $\omega = \sqrt{3}$, $H = 4$, $c = 2$, $m = 1$.



4/14/20 — Follow notes of 10/18/19, 10/21/19, 10/23/19, Linear differential equation of order n, Constant coefficient homogeneous linear equations (homogeneous)

4/16/20 — Follow notes of 10/25/19, 10/28/19, 11/1/19 Undetermined coefficients and annihilators, Variation of parameters for order n.