4/7/20 — Variation of parameters — second order equations

Consider $y'' + ay' + by = F(x)$. Suppose $c_1y_1 + c_2y_2$ solves the homogeneous equation $y'' + ay' + by = 0$. Set $y = u_1y_1 + u_2y_2$, where $u_1 = u_1(x)$ and $u_2 = u_2(x)$. Want to find u_1 and u_2 to solve the given inhomogeneous problem.

Calculate $y' = u_1'y_1 + u_2'y_2 + u_1y_1' + u_2y_2'$. Suppose side condition $u_1'y_1 + u_2'y_2 = 0$, so $y' = u_1y_1' + u_2y_2'$. Differentiate again $y'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$. Substitute into the original problem:

$$F = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'' + a(u_1y_1' + u_2y_2') + b(u_1y_1 + u_2y_2)$$

$$= u_1'y_1' + u_2'y_2' + u_1(y_1'' + ay_1' + by_1) + u_2(y_2'' + ay_2' + by_2)$$

Since $y_1'' + ay_1' + by_1 = 0$ and $y_2'' + ay_2' + by_2 = 0$, we need only solve $y_1u_1' + y_2u_2' = 0$ and $y_1'u_1' + y_2'u_2' = F$. Equivalently

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ F \end{pmatrix}$$

It follows from the linear independence of y_1 and y_2 that the Wronskian

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1' \neq 0, \text{ for all } x.$$

So one can solve for u_1' and u_2' by Cramer's rule. Integration yields u_1 and u_2. In summary, for any $F(x)$, we obtain a particular solution of $y'' + ay' + by = F(x)$.

Note that the method of undetermined coefficients only works for constant a, b and a limited collection of $F(x)$.

Do examples p. 512/6, 2

<u>Example 1</u> $y'' + 9y = 18\sec^3(x\cdot 3)$, $|x| < \frac{\pi}{6}$

Start by solving $y'' + 9y = 0$, $r^2 + 9 = 0$, $r = \pm 3i$, so $y = c_1 \cos 3x + c_2 \sin 3x$. Next proceed with variation of parameters:

$$A = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{pmatrix}, \quad A^{-1} = \frac{1}{3}\begin{pmatrix} 3\cos 3x & -\sin 3x \\ 3\sin 3x & \cos 3x \end{pmatrix}$$

Thus

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ 18\sec^3(3x) \end{pmatrix} = \begin{pmatrix} -\sec^3(3x)\cdot \sin 3x \\ \sec^3(3x)\cdot \cos 3x \end{pmatrix}\cdot 6$$

Moreover $u_1 = -6\int \tan 3x \sec^2(3x)\,dx = -2\int \tan 3x\, d(\tan 3x)$
$= -\tan^2(3x)$ and $u_2 = \int \sec^3 3x \cos 3x\, dx = 6\int \sec^2 3x\, dx$
$= 2\tan 3x$

So $y_p = u_1 y_1 + u_2 y_2 = (-\tan^2 3x)\cos 3x + 2(\tan 3x)\sin 3x$

$$= \frac{-\sin^2 3x}{\cos 3x} + \frac{2\sin^2 3x}{\cos 3x} = \frac{\sin^2 3x}{\cos 3x} = \frac{1 - \cos^2 3x}{\cos 3x}$$

Thus $y_p = \sec 3x - \cos 3x$. Since $\cos 3x$ solves the homogeneous problem $y_p = \sec 3x$ suffices.

The general solution is $y = \sec 3x + c_1 \cos 3x + c_2 \sin 3x$

<u>Remark</u> One requires $|x| < \frac{\pi}{6}$ to avoid the singularity of $\sec 3x$ when $|x| = \pi/6$.

Example 2 $y'' - 4y = \dfrac{8}{e^{2x}+1}$

The solution of the homogeneous problem is $c_1 e^{2x} + c_2 e^{-2x}$.
So the variation of parameters reads

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 8(e^{2x}+1)^{-1} \end{pmatrix} \text{ where } A = \begin{pmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{pmatrix}$$

Thus $$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \frac{-1}{4} \begin{pmatrix} -8\, e^{-2x}(e^{2x}+1)^{-1} \\ 8\, e^{2x}\, (e^{2x}+1)^{-1} \end{pmatrix}$$

Integrating gives $u_1 = \displaystyle\int \frac{2\,dx}{e^{2x}(e^{2x}+1)} = 2\int\left(\frac{dx}{e^{2x}} - \frac{dx}{e^{2x}+1}\right)$

$= -e^{-2x} + 2\displaystyle\int\left(\frac{e^{2x}}{e^{2x}+1} - 1\right)dx = -e^{-2x} + \ln(e^{2x}+1) - 2x$

and $u_2 = -2\displaystyle\int \frac{e^{2x}\,dx}{e^{2x}+1} = -\ln(e^{2x}+1)$

So $y_p = u_1 y_1 + u_2 y_2 = \left(-e^{-2x} + \ln(e^{2x}+1) - 2x\right)e^{2x}$

$- \ln(e^{2x}+1)\, e^{-2x} = \ln(e^{2x}+1)(e^{2x} - e^{-2x}) - 2x e^{2x} - 1$

Remark Alternatively one can reduce the problem by factoring $D^2 - 4 = (D+2)(D-2)$, where $D = \frac{d}{dx}$.
First solve
$$(D+2)f = \frac{8}{e^{2x}+1}, \quad f = 4e^{-2x}\ln(e^{2x}+1)$$
and then
$$(D-2)g = f = 4e^{-2x}\ln(e^{2x}+1)$$
These are both first order linear equations.
After a somewhat lengthy calculation one again obtains the ~~same~~ result given by variation of parameters.

The general solution is $y = \ln(e^{2x}+1)(e^{2x} - e^{-2x}) - 2x e^{2x} - 1$
$+ c_1 e^{2x} + c_2 e^{-2x}$

11/6/19 — Reduction of order — Second order

Given $y'' + a_1(x)y' + a_2(x)y = F(x)$ and y_1 solves homogeneous, i.e. $F(x) = 0$.
Substitute $y(x) = u(x)y_1$, $y' = u'y_1 + uy_1'$, $y'' = u''y_1 + 2u'y_1' + uy_1''$. So $y'' + a_1(x)y' + a_2(x)y = y_1u'' + (a_1y_1 + 2y_1')u' = F(x)$. First order linear for u', solve to get general solution of original problem, using method for first order linear equations.

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<u>Remark</u> Method is simple enough to be applied directly from $y(x) = u(x)y_1$, without memorization of final formula. Compare variation of parameters.

<u>Remark</u> For 9(c), define $y_\beta(x) = \left(\frac{-1}{\beta!}\right) e^{+x} \int_{-\infty}^{x} x^\beta e^{-x}\, dx$. Use induction of β and partial integration. Else check by direct calculation from formula given in 9(c).

Reduction of order — Follow notes of 11/6/19 —

Example 3 $x^2y'' - 3xy' + 4y = 0$, $x > 0$. Given that $y_1(x) = x^2$ is a solution. Find a second solution.

Renormalize $y'' - 3x^{-1}y' + 4x^{-2}y = 0$, to compare with model $y'' + a_1 y' + a_2 y = 0$. So $a_1 = -3x^{-1}$ and $a_2 = 4x^{-2}$. If $y_2 = uy_1 = ux^2$, then we have $x^2u'' + u'[4x - 3x] = 0$, $u'' + x^{-1}u' = 0$. This is first order in $p = u'$, so $p = -x^{-1}$ and $u = -\ln x$. So $y_2(x) = x^2 \ln x$. The general solution is $c_1x^2 + c_2x^2\ln x$. Note that the condition $x > 0$ is imposed to avoid the singularity of x^{-1} at $x = 0$.

Example 4 $xy'' + (1-2x)y' + (x-1)y = 0$, $x > 0$. Given that $y_1 = e^x$ is a solution. Find a second solution.

Renormalize $y'' + (x^{-1} - 2) + (1 - x^{-1})y = 0$, so $a_1 = x^{-1} - 2$ and $a_2 = 1 - x^{-1}$. If $y_2 = e^x u$, then by comparison with the general derivation, $u'' + x^{-1}u' = 0$, $u = \ln x$, and $y = e^x \ln x$. The general solution is $c_1e^x + c_2e^x\ln x$.

Example 5 $y'' - 4y' + 4y = 4e^{2x}\ln x$. Given that $y_1 = e^{2x}$ solves the homogeneous problem, find the general solution.

Since the equation has constant coefficients $y_2 = xe^{2x}$ is a second solution to the homogeneous. So we only need to find a particular solution to the inhomogeneous. Here $a_1 = -4$ and $a_2 = 4$. Suppose $y_p = ue^{2x}$. Then $e^{2x}u'' = 4e^{2x}\ln x$, $u'' = 4\ln x$. Integrating twice gives $u = 2x^2\ln x - 3x^2$, so the general solution is $c_1e^{2x} + c_2xe^{2x} + (2x^2\ln x - 3x^2)e^{2x}$