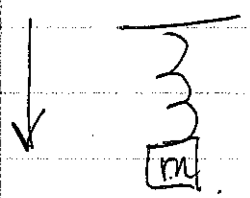
Section 4.1

Mass Spring Oscillator

Mass m attached to spring fixed at one end. y. displacement



- Hooke's law

$$F_s = -ky \quad k \geq 0$$

- Friction

$$F_f = -by' \quad b \geq 0$$

Other forces (gravitational, electrical etc...)

Newton's 2^{nd} law of motion

$$my'' = -ky - by' + F_{ext}$$

$$my'' + by' + ky = F_{ext}$$

$b = 0 \iff$ no friction

$F_{ext} = 0$

look for solutions in the form: $y = \cos \omega t$

$$y' = -\omega \sin \omega t$$

$$y'' = -\omega^2 \cos \omega t$$

$my'' + \cancel{\gamma y'} + ky = 0 \qquad y = \cos \omega t$

$-m\omega^2 \cancel{\cos \omega t} + k \cancel{\cos \omega t} = 0$

$k = +m\omega^2 \iff \omega = \sqrt{k/m}$

Ex. $\underline{y'' + 2y' + 5y = -50 \sin 5t}$

Find A, B so that y is solution $\longrightarrow$ $\boxed{y = A \cos 5t + B \sin 5t}$

$y' = -5A \sin 5t + 5B \cos 5t$

$y'' = -25A \cos 5t - 25B \sin 5t$

$-25A \cos 5t - 25B \sin 5t - 10A \sin 5t$
$+ 10B \cos 5t + 5A \cos 5t + 5B \sin 5t$
$= -50 \sin 5t$

$\underline{\cos 5t}$ $\quad -25A + 5A \cancel{-10} + 10B = 0$

$\underline{\sin 5t}$ $\quad -25B - 10A + 5B = -50$

$$\begin{cases} -20A + 10B = 0 \\ -20B - 10A = -50 \end{cases}$$

$2B + A = 5$

$$\begin{cases} B = 2A \\ 4A + A = 5 \end{cases}$$

$A = 1$

$B = 2$

$y = \cos 5t + 2 \sin 5t$

Sect. 4.2

$$\boxed{ay'' + by' + cy = 0}$$

look for solution in form $y = e^{rt}$

Compute derivatives $y' = re^{rt} \quad y'' = r^2 e^{rt}$

Replace in equ. $ar^2\cancel{e^{rt}} + br\cancel{e^{rt}} + c\cancel{e^{rt}} = 0$

$$\boxed{ar^2 + br + c = 0}$$

auxiliary (characteristic) equation

Two roots r_1, r_2

Two corr. solutions $y_1 = e^{r_1 t}$

$y_2 = e^{r_2 t}$

General solution is given by

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Ex $2y'' + 7y' - 4y = 0$

$2r^2 + 7r - 4 = 0$

$$r = \frac{-7 \pm \sqrt{49+32}}{4} = \frac{-7 \pm \sqrt{81}}{4}$$

$$= \frac{-7 \pm 9}{4} = \begin{cases} 1/2 \\ -4 \end{cases}$$

$y_1 = e^{\frac{1}{2}t} \qquad y_2 = e^{-4t}$

$$y = c_1 e^{\frac{1}{2}t} + c_2 e^{-4t}$$

Ex. $y'' + 5y' + 6y = 0$

$r^2 + 5r + 6 = 0 \qquad r = -2, -3$

$y_1 = e^{-2t} \qquad y_2 = e^{-3t}$

$y = c_1 e^{-2t} + c_2 e^{-3t}$

Ex. $y'' + 6y' + 9 = 0$

$r^2 + 6r + 9 = 0 \qquad (r+3)^2 = 0$

$r_1 = r_2 = -3$

$y_1 = e^{-3t} \qquad y_2 = ?$

Take $y_2 = t e^{-3t}$

$y = c_1 e^{-3t} + c_2 t e^{-3t}$

Thm. If $ar^2 + br + c = 0$ has two coincident roots $r_1 = r_2 = r$, then two solutions are given by $y_1 = e^{rt}$, $y_2 = te^{rt}$

and gen. sol is $y = c_1 e^{rt} + c_2 t e^{rt}$.

Ex. $4y'' - 4y' + y = 0$

$$4r^2 - 4r + 1 = 0$$

$$(2r-1)^2 = 0 \qquad r = \tfrac{1}{2}$$

$$y_1 = e^{\frac{1}{2}t} \qquad y_2 = te^{\frac{1}{2}t}$$

$$y = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$$

Ex. Initial Value Problem (IVP)

$$y'' + 2y' - 8y = 0 \qquad y(0) = 3$$
$$y'(0) = -12$$

$$r^2 + 2r - 8 = 0 \qquad r_1 = 2$$
$$r_2 = -4$$

$$y_1 = e^{2t} \qquad y_2 = e^{-4t}$$

gen sol. $\rightarrow$ $y = c_1 e^{2t} + c_2 e^{-4t}$

$$y' = 2c_1 e^{2t} - 4c_2 e^{-4t}$$

$$\begin{cases} 3 = y(0) = c_1 + c_2 \\ -12 = y'(0) = 2c_1 - 4c_2 \end{cases}$$

$$\begin{cases} c_1 + c_2 = 3 \\ 2c_1 - 4c_2 = -12 \end{cases}$$

$$\begin{cases} c_1 = 3 - c_2 \\ 2(3 - c_2) - 4c_2 = -12 \end{cases}$$

$$\begin{cases} c_1 = 3 - c_2 \\ 6 - 2c_2 - 4c_2 = -12 \end{cases}$$

$$-6c_2 = -18$$

$$c_2 = 3$$

$$c_1 = 3 - c_2 = 3 - 3 = 0$$

$$y = c_1 e^{2t} + c_2 e^{-4t}$$

$$\boxed{y = 3e^{-4t}}$$

Check solution is correct: $y' = -12e^{-4t}$ $\quad y'' = 48e^{-4t}$

$$y'' + 2y' - 8y = 0$$

$$48e^{-4t} - 24e^{-4t} - 24e^{-4t} = 0 \checkmark$$

$$y(0) = 3 \checkmark$$

$$y'(0) = -12 \checkmark$$

Def. Two functions y_1 & y_2 are LD (linearly dependent) on some interval I if they are one a constant multiple of the other on I

Ex. $y_1 = t^2 + 3t \qquad y_2 = 2t^2 + 6t$

$$y_2 = 2(t^2 + 3t)$$
$$= 2y_1$$

y_1, y_2 LD

Ex. $y_1 = t^2 + 3t \qquad y_2 = 2t^2 + 8t$

y_1 ~~not~~ multiple of y_2

y_1 & y_2 are not LD

linearly independent (LI)

Ex. $y_1 = \cos t \sin t \qquad y_2 = 2 \sin t$

LI

Ex. $y_1 = 3 \sin t \qquad y_2 = 2 \sin t$

LD

Ex. $y_1 = \cos t \sin t \qquad y_2 = 2 \sin 2t$

$y_2 = 2 \cos t \sin t \qquad y_2 = 2 y_1$

LD

Ex. $y_1 = e^{4t}$ $\qquad y_2 = \frac{3}{2}e^{4t}$

$$y_2 = \frac{3}{2} y_1 \qquad \text{LD}$$

Ex. $y_1 = e^{4t}$ $\qquad y_2 = te^{4t}$

$$y_2 = t\, y_1 \qquad t \text{ not const.}$$

$$\text{LI}$$

Ex. $y_1 = e^{2t}$ $\qquad y_2 = e^{4t}$

$$y_1 \neq k y_2 \qquad \text{LI}.$$

Complex roots   Section 4.3

Ex. $y'' - 10y' + 26y = 0$

$r^2 - 10r + 26 = 0$ characteristic (auxiliary) eq^n.

$r = 5 \pm \sqrt{25-26} = 5 \pm \sqrt{-1}$
$= 5 \pm i$

Property: If $ar^2 + br + c = 0$ has complex roots $\alpha \pm i\beta$, then general solution to

$$ay'' + by' + cy = 0$$

~~...~~ is

$$y = c_1 \underbrace{e^{\alpha t}\cos\beta t}_{y_1} + c_2 \underbrace{e^{\alpha t}\sin\beta t}_{y_2}$$

Ex. $\alpha = 5 \qquad \beta = 1$

$$y = c_1 e^{5t}\cos t + c_2 e^{5t}\sin t$$

Ex. $4y'' - 4y' + 26y = 0$

$$4r^2 - 4r + 26 = 0$$

$$r = \frac{2 \pm \sqrt{4-104}}{4} = \frac{2 \pm \sqrt{-100}}{4}$$

$$= \frac{2 \pm 10i}{4}$$

$$= \underset{\alpha}{\frac{1}{2}} \pm \underset{\beta}{\frac{5}{2}} i$$

$$y = c_1 e^{\frac{1}{2}t} \cos \frac{5}{2}t + c_2 e^{\frac{1}{2}t} \sin \frac{5}{2}t$$

Ex. $u'' + 16u = 0$

$$r^2 + 16 = 0 \qquad r^2 = -16$$

$$r = \pm\sqrt{-16} = \pm 4i$$

$$\alpha = 0 \qquad \beta = 4$$

$$y_1 = c_1 \cancel{e^{0t}} \cos 4t + c_2 \cancel{e^{0t}} \sin 4t$$

$$= c_1 \cos 4t + c_2 \sin 4t$$

Rmk $\alpha = 0 \rightarrow$ gen sol. is

$$y = c_1 \cos \beta t + c_2 \sin \beta t$$

IVP

Ex. $y'' + 2y' + 17y = 0$ $\quad y(0) = 1$, $\quad y'(0) = -1$.

$r^2 + 2r + 17 = 0$

$r = -1 \pm \sqrt{1 - 17} = -1 \pm 4i$

$\alpha = -1 \quad \beta = 4$

$y = c_1 e^{-t} \cos 4t + c_2 e^{-t} \sin 4t$

$1 = y(0) = c_1 \Rightarrow c_1 = 1$

$\boxed{y = e^{-t} \cos 4t + c_2 e^{-t} \sin 4t}$

$= e^{-t} (\cos 4t + c_2 \sin 4t)$

$y' = -e^{-t} (\cos 4t + c_2 \sin 4t)$
$+ e^{-t} (-4 \sin 4t + 4 c_2 \cos 4t)$

$-1 = y'(0) = -1 + 4c_2$

$4c_2 - 1 = -1$

$c_2 = 0$

$\boxed{y = e^{-t} \cos 4t}$

Ex. $y'' + 9y = 0$ $\quad y(0) = 1$, $\quad y'(0) = 1$

$r^2 + 9 = 0 \qquad r^2 = -9 \qquad r = \pm 3i$

$\alpha = 0 \quad \beta = 3$

$y = c_1 \cos 3t + c_2 \sin 3t$

$1 = y(0) = c_1 \qquad c_1 = 1$

$y = \cos 3t + c_2 \sin 3t$

$y' = -3 \sin 3t + 3 c_2 \cos 3t$

$1 = y'(0) = 3c_2 \rightarrow c_2 = 1/3$

$y = \cos 3t + \frac{1}{3} \sin 3t$

Ex. $y'' + by' + 4y = 0 \qquad y(0) = 1, \quad y'(0) = 0$

① $b = 5$

$y'' + 5y' + 4y = 0$

$r^2 + 5r + 4 = 0$

$r = -1, -4$

$y_1 = e^{-t} \qquad y_2 = e^{-4t}$

$y = c_1 e^{-t} + c_2 e^{-4t}$ gen. sol.

$y' = -c_1 e^{-t} - 4c_2 e^{-4t}$

$$\begin{cases} 1 = y(0) = c_1 + c_2 \\ 0 = y'(0) = -c_1 - 4c_2 \end{cases}$$

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 - 4c_2 = 0 \end{cases} \qquad \begin{array}{l} -4c_2 + c_2 = 1 \\ c_1 = -4c_2 \end{array}$$

$-3c_2 = 1 \rightarrow c_2 = -1/3$

$c_1 = -4c_2 = 4/3$

$y = 4/3\, e^{-t} - \frac{1}{3} e^{-4t}$

Plot the graph

$y'' + by' + 4y = 0 \qquad y(0) = 1$

$\qquad\qquad\qquad\qquad\quad y'(0) = 0$

(2) $b = 4$

$y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0 \qquad (r+2)^2 = 0$

$r_1, r_2 = -2$

$y = c_1 e^{-2t} + c_2 t e^{-2t}$ gen sol.

$y' = -2c_1 e^{-2t} + c_2 e^{-2t}(1 + (-2)t)$

$y' = -2c_1 e^{-2t} + c_2 e^{-2t}(1 - 2t)$

$1 = y(0) = c_1$

$0 = y'(0) = -2 + c_2 \rightarrow c_2 = 2$

$y = e^{-2t} + 2te^{-2t}$

Plot the graph

$y'' + by' + 4y = 0$ $\qquad y(0) = 1$, $\quad y'(0) = 0$

3 $\quad b = 2$

$$y'' + 2y' + 4y = 0$$

$$r^2 + 2r + 4 = 0$$

$$r = -1 \pm \sqrt{1-4} = -1 \pm \sqrt{-3}$$
$$= -1 \pm i\sqrt{3}$$

gen. sol. $\rightarrow$ $y = c_1 e^{-t} \cos\sqrt{3}t + c_2 e^{-t} \sin\sqrt{3}t$
$$= e^{-t}(c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t)$$

$$1 = y(0) = c_1$$

$$y' = -e^{-t}(c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t) + e^{-t}(-c_1\sqrt{3} \sin\sqrt{3}t + \sqrt{3}c_2 \cos\sqrt{3}t)$$

$$0 = y'(0) = -c_1 + \sqrt{3}c_2$$
$$= -1 + c_2\sqrt{3}$$

$$c_2\sqrt{3} = 1 \qquad c_2 = \frac{\sqrt{3}}{3}$$

$$y = e^{-t}\cos\sqrt{3}t + \frac{\sqrt{3}}{3} e^{-t} \sin\sqrt{3}t$$

Plot the graph