Section 4.4

Method of undetermined coefficients

$$ay'' + by' + cy = 0 \quad \text{homogeneous eqn.}$$

$$ay'' + by' + cy = f(t) \quad \text{non-homogeneous eqn.}$$

Method works only for certain types of RHS $f(t)$

$f(t) =$
- exponential
- polynomials
- sin / cos

or products of the above.

<u>Ex.</u> $2y'' + y = \underline{\underline{9e^{2t}}}$

$\rightarrow y_p = Ae^{2t}$

$\uparrow$ undetermined coeff.

$$y_p' = 2Ae^{2t} \qquad y_p'' = 4Ae^{2t}$$

$$8Ae^{2t} + Ae^{2t} = 9e^{2t}$$

$$9A\cancel{e^{2t}} = 9\cancel{e^{2t}} \qquad \underline{A = 1}$$

$$y_p = e^{2t}$$

Ex.

$$y'' - 4y = 9e^{2t}$$

$$y_p = Ae^{2t} \qquad y_p' = 2Ae^{2t} \qquad y_p'' = 4Ae^{2t}$$

$$4Ae^{2t} - 4Ae^{2t} = 9e^{2t}$$

$$\cancel{0 = 9e^{2t}}$$

$$y'' - 4y = 0$$

$$r^2 - 4 = 0 \qquad r^2 = 4 \qquad r = \pm 2$$

$$y_1 = e^{2t} \qquad y_2 = e^{-2t}$$

$$\cancel{y_p = Ae^{2t}} \qquad y_p = Ate^{2t} \leftarrow$$

$$y_p' = Ae^{2t}(1 + 2t)$$

$$y_p'' = Ae^{2t}(2(1+2t) + 2)$$

$$= Ae^{2t}(4 + 4t)$$

$$y'' - 4y = 9e^{2t}$$

Plug in

$$A\cancel{e^{2t}}(4+4t) - 4At\cancel{e^{2t}} = 9\cancel{e^{2t}}$$

$$4A + \cancel{4At} - \cancel{4At} = 9$$

$$A = 9/4$$

$$\boxed{y_p = \tfrac{9}{4} t e^{2t}}$$

Ex. $y'' + 3y = -9$

$y_p = A \qquad y_p' = 0 \qquad y_p'' = 0$

$3A = -9 \qquad A = -3$

$$\boxed{y_p = -3}$$

Ex. $y'' + 3y' = \underline{\underline{-9}} \leftarrow$

$y_p = A \qquad y_p' = y_p'' = 0$

$\cancel{0 \neq -9}$

$y'' + 3y' = 0$

$r^2 + 3r = 0 \qquad r(r+3) = 0$

$r_1 = 0 \qquad r_2 = -3$

$y_1 = e^{0t} = 1 \qquad y_2 = e^{-3t}$

$y_p = At$

$y_p' = A \qquad y_p'' = 0$

$+3A = -9 \qquad A = -3$

$y_p = -3t$

Ex. $2y'' + y = 3t^2$

$\cancel{y_p = At^2}$ $\quad y_p' = 2At \quad y_p'' = 2A$

$4A + At^2 = 3t^2$

t^2 $\quad A = 3 \qquad\qquad A = 3$

$t^0 \quad 4A = 0 \qquad\qquad A = 0$

contr.

$y_p = At^2 + Bt + C$

$y_p' = 2At + B$

$y_p'' = 2A$

$4A + At^2 + Bt + C = 3t^2$

t^2 $\quad A = 3$

t $\quad B = 0$

t^0 $\quad 4A + C = 0 \qquad C = -4A = -12$

$y_p = 3t^2 - 12$

Ex. $y'' - y' + 9y = 3\sin 3t$

$$\cancel{y_p = A\sin 3t}$$

$$y_p' = 3A\cos 3t$$

$$y_p'' = -9A\sin 3t$$

$$\cancel{-9A\sin 3t} - 3A\cos 3t + \cancel{9A\sin 3t} = 3\sin 3t$$

$$y_p = A\sin 3t + B\cos 3t \leftarrow$$

$$y_p' = 3A\cos 3t - 3B\sin 3t$$

$$y_p'' = -9A\sin 3t - 9B\cos 3t$$

$$-9A\sin 3t - 9B\cos 3t - 3A\cos 3t + 3B\sin 3t + 9A\sin 3t + 9B\cos 3t = 3\sin 3t$$

sin $\quad -\cancel{9A} + 3B + \cancel{9A} = 3 \qquad B = 1$

cos $\quad -\cancel{9B} - 3A + \cancel{9B} = 0 \qquad A = 0$

$$y_p = \cos 3t$$

Ex. $\underline{\underline{y'' + 4y}} = 16 \sin 2t$

$y_p = A \sin 2t + B \cos 2t$

$y_p' = 2A \cos 2t - 2B \sin 2t$

$y_p'' = \cancel{-4A \cos 2t - 4B \cos}$

$-4A \sin 2t - 4B \cos 2t$

$\cancel{-4A \sin 2t} - \cancel{4B \cos 2t} + \cancel{4A \sin 2t}$

$+ \cancel{4B \cos 2t} = 16 \sin 2t$

$0 = 16 \sin 2t$

$y'' + 4y = 0$

$r^2 + 4 = 0$

$r^2 = -4$

$r = \pm 2i$

$\alpha \pm i\beta \qquad \alpha = 0$

$y_1 = \cos 2t$

$y_2 = \sin 2t$

$y = c_1 \cos 2t + c_2 \sin 2t$

$y_p = t(A \sin 2t + B \cos 2t)$

$y_p' = A \sin 2t + B \cos 2t$

$+ 2tA \cos 2t - 2tB \sin 2t$

$$y_p'' = 2A\cos 2t - 2B\sin 2t$$
$$+ 2A\cos 2t - 4tA\sin 2t$$
$$- 2B\sin 2t - 4tB\cos 2t$$

$$y_p'' = 4A\cos 2t - 4B\sin 2t$$
$$- 4tA\sin 2t - 4tB\cos 2t$$

$$y'' + 4y = 16\sin 2t$$
$$y_p = t(A\sin 2t + B\cos 2t)$$

$$-4A\cos 2t - 4B\sin 2t - \cancel{4tA}\sin 2t$$
$$+ \cancel{4tB}\cos 2t$$
$$+ 4\cancel{At}\sin 2t - 4t\cancel{B}\cos 2t$$
$$= 16\sin 2t$$

$\cos 2t$	$-4A = 0$	$A = 0$
$\sin 2t$	$-4B = 16$	$B = -4$

$$y_p = -4t\cos 2t$$

Ex. $y'' - 5y' + 4y = te^{2t}$

$$y_p = (At + B)\,e^{2t}$$

$$y_p' = (A + 2At + 2B)\,e^{2t}$$

$$y_p'' = (2A + 2A + 4At + 4B)\,e^{2t}$$

$$= (4A + 4At + 4B)\,e^{2t}$$

$$(4A + 4At + 4B)\cancel{e^{2t}} - 5(A + 2At + 2B)\cancel{e^{2t}} + 4(At + B)\cancel{e^{2t}} = t\cancel{e^{2t}}$$

$$4A + 4At + 4B - 5A - 10At - 10B + 4At + 4B = t$$

$$4A + 4B - 5A - 10B + 4B = 0$$

coeff t

$$4A - 10A + 4A = 1$$

$$-2A = 1 \qquad A = -\tfrac{1}{2}$$

$$-A - 2B = 0$$

$$2B = -A = \tfrac{1}{2}$$

$$B = \tfrac{1}{4}$$

$$y_p = (At+B)e^{2t}$$

$$A = -\tfrac{1}{2} \qquad B = \tfrac{1}{4}$$

$$y_p = \left(-\tfrac{1}{2}t + \tfrac{1}{4}\right)e^{2t}$$

Ex. $y'' - 4y' + 4y = te^{2t}$

$$y_p = (At+B)e^{2t}$$

$$y_p' = (A + 2At + 2B)e^{2t}$$

$$y_p'' = (4A + 4At + 4B)e^{2t}$$

$$(4A + 4At + 4B)\cancel{e^{2t}} - 4(A + 2At + 2B)\cancel{e^{2t}} + 4(At+B)\cancel{e^{2t}} = t\cancel{e^{2t}}$$

$$\cancel{4A} + \cancel{4At} + \cancel{4B} - \cancel{4A} - \cancel{8At} - \cancel{8B} + \cancel{4At} + \cancel{4B} = t$$

$$\cancel{0 = t}$$

$$y'' - 4y' + 4y = 0$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0 \qquad r = 2$$

$$y_1 = e^{2t} \qquad y_2 = te^{2t}$$

Try $y_p = t(At+B)e^{2t} = (At^2+Bt)e^{2t}$

$$y_p' = (2At + B + 2At^2 + 2Bt)e^{2t}$$

$$y_p'' = (2A + 4At + 2B + 4At + 2B + 4At^2 + 4Bt)e^{2t}$$

$\cancel{2A + 4At + 2B + 4At +}$

$$(2A + 8At + 4B + 4At^2 + 4Bt)\cancel{e^{2t}}$$
$$- (8At + 4B + 8At^2 + 8Bt)\cancel{e^{2t}}$$
$$+ (4At^2 + 4Bt)\cancel{e^{2t}} = te^{2t}$$

$2A = t$ not possible!

Since $r = 2$ is double root, we need to multiply by one additional factor of t, that is

$$y_p = t^2(At+B)e^{2t}$$

Leave it to you to compute

$$y_p', \; y_p''$$

Plug in eqn., find A & B.

Rules

① $f(t) = Ct^m e^{rt}$

$$y_p(t) = t^s(A_1 t^m + A_2 t^{m-1} + \dots + A_{m+1}) e^{rt}$$

i) $s=0$ if r not root of char. eqn.

ii) $s=1$ if r simple root

iii) $s=2$ if r double root

② $f(t) = Ct^m e^{\alpha t} \cos \beta t$

$= Ct^m e^{\alpha t} \sin \beta t$

$$y_p(t) = t^s(A_1 t^m + A_2 t^{m-1} + \dots + A_{m+1}) e^{\alpha t} \cdot \cos \beta t$$

$$+ t^s(B_1 t^m + B_2 t^{m-1} + \dots + B_{m+1}) e^{\alpha t} \sin \beta t$$

i) $s=0$ if $\alpha \pm i\beta$ not root

ii) $s=1$ if $\alpha \pm i\beta$ root

Ex. $y'' + 9y = 4t^3 \sin 3t$

$c = 4$

$m = 3$

$\beta = 3$

$\alpha = 0$

$\alpha \pm i\beta$ root

$\pm 3i$ root ?

$y'' + 9y = 0$

$r^2 + 9 = 0 \qquad r^2 = -9$

$r = \pm 3i$ yes

$$y_p = t\{(A_1t^3 + A_2t^2 + A_3t + A_4)\sin 3t\}$$
$$+ t\{(B_1t^3 + B_2t^2 + B_3t + B_4)\cos 3t\}$$