Ex. $y'' - 6y' + 9y = 5t^6 e^{3t}$

$y_p = ?$

$r^2 - 6r + 9 = 0 \qquad r = 3 \qquad (r-3)^2 = 0$

double root

$\cancel{y_p = A_1 t^6 + A_2 t^5 + A_3 t^4 + A_4 t^5 + A}$

$y_p = t^2 (A_1 t^6 + A_2 t^5 + A_3 t^4 + A_4 t^3 + A_5 t^2$
$+ A_6 t + A_7) e^{3t}$

Ex. $y''' - y'' + y = \underline{\underline{\sin t}}$

roots $\alpha \pm i\beta$

$y_1 = e^{\alpha t} \cos \beta t \qquad y_2 = e^{\alpha t} \sin \beta t$

$\left.\begin{array}{l} \alpha = 0 \\ \beta = 1 \end{array}\right\} \rightarrow \sin t$

$r = \pm i$

Is $\pm i$ root of char. eqn?

$r^3 - r^2 + 1 = 0$ char. eqn.

Check if i root plug in

$i^3 - i^2 + 1 \stackrel{?}{=} 0 \qquad -i + 1 + 1 = 2 - i \neq 0$

$\Rightarrow \pm i$ not roots.

$y_p = A\cos t + B\sin t$

$y_p' =$

$y_p'' =$

Plug into eqn.

Section 4.5

Superposition principle

Thm. y_1 solution to $ay'' + by' + cy = f_1(t)$

y_2 solution to $ay'' + by' + cy = f_2(t)$

$\Rightarrow$ $k_1 y_1 + k_2 y_2$ solution to

$ay'' + by' + cy = k_1 f_1(t) + k_2 f_2(t)$

k_1, k_2 constants

Pf. Immediate. Leave as exercise

Ex. $y_1 = \cos t$ sol. to $y'' - y' + y = \sin t$

$y_2 = \frac{e^{2t}}{3}$ sol. to $y'' - y' + y = e^{2t}$

$y'' - y' + y = \sin t \longrightarrow$ $r = \pm i$ not roots

$r^2 - r + 1 = 0$ char. eqn.

$i^2 - i + 1 = -1 - i + 1 = -i \neq 0$

$y'' - y' + y = \sin t \longrightarrow \quad r = \pm i$

$y_p = (A \sin t + B \cos t)$

$y_p' = A \cos t - B \sin t$

$y_p'' = -A \sin t - B \cos t$

$\cancel{-A \sin t} - \cancel{B \cos t} - A \cos t + B \sin t$
$+ \cancel{A \sin t} + \cancel{B \cos t} = \sin t$

$\underline{\cos t} \quad -A = 0 \qquad A = 0$

$\underline{\sin t} \quad B = 1$

$y_p = \cos t$

$y'' - y' + y = e^{2t}$

$y_p = \frac{1}{3} e^{2t} \leftarrow$

$y_p = A e^{2t}$ $\qquad e^{rt}$ sol. to hom. eqn. $\quad r = 2$

$r^2 - r + 1 = 0$

$4 - 2 + 1 \neq 0$

$y_p' = 2A e^{2t} \qquad y_p'' = 4A e^{2t}$

$4A \cancel{e^{2t}} - 2A \cancel{e^{2t}} + A \cancel{e^{2t}} = \cancel{e^{2t}}$,

$3A = 1 \qquad A = 1/3$

$$y'' - y' + y = \sin t = f_1 \qquad y_1 = \cos t$$

$$y'' - y' + y = e^{2t} = f_2 \qquad y_2 = \frac{e^{2t}}{3}$$

1. $y'' - y' + y = \underline{5} \sin t \qquad y = \underline{5} \cos t$

2. $y'' - y' + y = \sin t - 3e^{2t} = f_1 - 3f_2$

$$y = y_1 - 3y_2 = \cos t - 3\frac{e^{2t}}{3}$$

$$= \cos t - e^{2t}$$

3. $y'' - y' + y = 4 \sin t + 18 e^{2t}$

$$= \cancel{4y_1 + 18 y_2} \; 4f_1 + 18 f_2$$

$$y = 4y_1 + 18 y_2 = 4 \cos t + 18 \frac{e^{2t}}{3}$$

$$= 4 \cos t + 6e^{2t}.$$

Construct general solution of non-homogeneous eqns.

$$ay'' + by' + cy = f(t) \neq 0$$

1. $ay'' + by' + cy = 0$

solution $c_1 y_1 + c_2 y_2 = y_c$

complementary solution

2. Find particular solution y_p (method undet. coeff.)

3. $y = c_1 y_1 + c_2 y_2 + y_p$

$= y_c + y_p$

general solution

c_1, c_2 arbitrary constant.

Ex. $y'' + y' = 1$ Find general sol.

$y_p = t$ (check!)

$y'' + y' = 0$

$r^2 + r = 0$

$r(r+1) = 0$

$r = 0, \quad r = -1$

$y_1 = e^{0t} = 1 \qquad y_2 = e^{-t}$

gen. sol. $y = c_1 \cdot 1 + c_2 \cdot e^{-t} + y_p$

$= c_1 + c_2 e^{-t} + t$

Ex. $y'' - 2y' + y = 2e^x$ $\quad y_p = x^2 e^x$ (check!)

$y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$ $\quad (r-1)^2 = 0$

$r = 1$ mult. 2

$y_1 = e^x$ $\quad y_2 = xe^x$

gen. sol. $y = c_1 y_1 + c_2 y_2 + y_p$

$= c_1 e^x + c_2 x e^x + x^2 e^x$

Ex. Find general sol.

$y'' - 2y' - 3y = 3t^2 - 5$

1 solve $y'' - 2y' - 3y = 0$

$r^2 - 2r - 3 = 0$

$r_1 = 3 \quad r_2 = -1$

$y_1 = e^{3t} \quad y_2 = e^{-t}$

$y_c = c_1 e^{3t} + c_2 e^{-t}$

2 $y'' - 2y' - 3y = 3t^2 - 5$

$y_p = At^2 + Bt + C \leftarrow$

$y_p' = 2At + B$

$y_p'' = 2A$

$2A - 4At - 2B - 3At^2 - 3Bt - 3C$

$= 3t^2 - 5$

t^2: $-3A = 3$ $A = -1$

t: $-4A - 3B = 0$

$3B = -4A = 4$

$B = 4/3$

t^0: $2A - 2B - 3C = -5$

$-2 - 8/3 - 3C = -5$

$-3C = -3 + 8/3 = -1/3$

$C = 1/9$

$y_p = -t^2 + \frac{4}{3}t + \frac{1}{9}$

3 Gen sol.

$$y = y_c + y_p$$
$$= c_1 e^{3t} + c_2 e^{-t} - t^2 + \frac{4}{3}t + \frac{1}{9}$$

__Ex.__ $y'' = 6t \qquad y(0) = 3 \qquad y'(0) = 1$

Find gen. sol.

1 $y'' = 0 \qquad r^2 = 0 \qquad r = 0$

$$y_1 = e^{0t} = 1 \qquad y_2 = t$$

$$y_c = c_1 + c_2 t$$

2 Find part. sol.

$$f(t) = 6t$$

$$y_p = t^2(At + B)$$
$$= At^3 + Bt^2 \leftarrow$$

$$y_p' = 3At^2 + 2Bt$$

$$y_p'' = 6At + 2B$$

$$6At + 2B = 6t$$

$$6A = 6 \qquad A = 1$$

$$2B = 0 \qquad B = 0$$

$$y_p = t^3$$

t
t^0

(3) gen ~~sol~~.

$$y = y_c + y_p$$
$$= c_1 + c_2 t + t^3$$

$y(0) = 3$ $\quad$ $y'(0) = 1$

$3 = y(0) = c_1$

$y' = c_2 + 3t^2$

$1 = y'(0) = c_2$

$$y = 3 + t + t^3$$
(check it is correct)

Ex. $y'' + y = 2e^{-x}$ $\quad$ $y(0) = 0$, $y'(0) = 0$

(1) $y'' + y = 0$ $\quad$ $r^2 + 1 = 0$

$r^2 = -1$

$r = \pm i$

$y_1 = \cos x$ $\quad$ $y_2 = \sin x$

$y_c = c_1 \cos x + c_2 \sin x$

2 Find y_p

$$y'' + y = 2e^{-x}$$

$$y_p = Ae^{-x} \leftarrow$$

$$y_p' = -Ae^{-x}$$

$$y_p'' = Ae^{-x}$$

$$A\cancel{e^{-x}} + A\cancel{e^{-x}} = 2\cancel{e^{-x}}$$

$$2A = 2 \rightarrow A = 1$$

$$\rightarrow y_p = e^{-x}$$

3 gen. sol.

$$y = y_c + y_p =$$

$$c_1 \cos x + c_2 \sin x + e^{-x} \leftarrow$$

4 $y(0) = 0 \quad y'(0) = 0$

$$0 = y(0) = c_1 + 1 \qquad c_1 = -1$$

$$y' = -c_1 \sin x + c_2 \cos x - e^{-x}$$

$$0 = y'(0) = c_2 - 1 \qquad c_2 = 1$$

$$y = -\cos x + \sin x + e^{-x}$$

Method of undetermined coeff. Part II

(1) $ay'' + by' + cy = P_m(t)\, e^{rt}$

$P_m(t)$ denotes polynomial of degree m

$$y_p = t^s(A_1 t^m + A_2 t^{m-1} + \dots + A_{m+1})\, e^{rt}$$

i) $s = 0$ if r not root of char. eqn.

ii) $s = 1$ if r simple root

iii) $s = 2$ if r double root

(2) $ay'' + by' + cy = P_m e^{\alpha t} \cos \beta t + Q_n e^{\alpha t} \sin \beta t$

P_m pol. of degree m

Q_n pol. of degree n

$$y_p = t^s(A_1 t^k + A_2 t^{k-1} + \dots + A_{k+1}) \cdot e^{\alpha t} \cos \beta t$$
$$+ t^s(B_1 t^k + B_2 t^{k-1} + \dots + B_{k+1}) \cdot e^{\alpha t} \sin \beta t$$

$k = \max\{m, n\}$

i) $s = 0$ if $\alpha \pm i\beta$ not root

ii) $s = 1$ if $\alpha \pm i\beta$ root.

Ex. Form of particular solution

$$y'' + y = \underbrace{\sin t + \cos t}_{f_1} + \underbrace{e^t}_{f_2}$$

eqn 1.

$$y'' + y = \sin t + \cos t$$

$$y_p^{(1)} = t(A \sin t + B \cos t) \leftarrow$$

$$y'' + y = 0$$

$$r^2 + 1 = 0 \qquad r = \pm i$$

$$y_1 = \cos t \qquad y_2 = \sin t$$

eqn 2.

$$y'' + y = e^t$$

$$y_p^{(2)} = C e^t \leftarrow$$

$$y_p = y_p^{(1)} + y_p^{(2)}$$

$$= t(A \sin t + B \cos t) + C e^t$$

Ex. Find form of particular sol.

$$y'' - y = e^{2t} + te^{2t} + t^2e^{2t}$$

$$= e^{2t}(\underbrace{1 + t + t^2}_{P_2(t)})$$

$$y_p = \cancel{t}\,(At^2 + Bt + C)e^{2t}$$

$$y'' - y = 0$$

$$r^2 - 1 = 0 \qquad r^2 = 1 \qquad r = \pm 1$$

$$y_1 = e^t \qquad y_2 = e^{-t}.$$