Method of variation of parameters

Solve non-homogeneous eqns

$$ay'' + by' + cy = f(t)$$

for <u>any</u> $f(t)$.

(1) Assume $a = 1$ (if not divide eqn. by a)

(2) Find complementary solution

$$y_c = c_1 y_1 + c_2 y_2,$$

where y_1 & y_2 are solutions to homogeneous eqn.

$$y'' + by' + cy = 0$$

(3) Find a particular solution in the form

$$y_p = u_1 y_1 + u_2 y_2$$

where $u_1 = u_1(t)$ and $u_2 = u_2(t)$ are functions, not constants.

To find u_1 & u_2, compute derivatives of y_p and plus into eqn

$$y'' + by' + cy = f(t).$$

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

Since we do not want second derivatives of u_1 & u_2, we impose

$$\boxed{u_1' y_1 + u_2' y_2 = 0}$$

Then $y_p' = u_1 y_1' + u_2 y_2'$

and $y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$

Plug in $y'' + by' + cy = f(t)$:

$$u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + b(u_1 y_1' + u_2 y_2')$$
$$+ c(u_1 y_1 + u_2 y_2) = f(t)$$

Rearranging terms

$$u_1 (y_1'' + by_1' + cy_1) + u_2 (y_2'' + by_2' + cy_2)$$
$$+ u_1' y_1' + u_2' y_2' = f(t)$$

But $y_1'' + by_1' + cy_1 = 0$ and

$y_2'' + by_2' + cy_2 = 0$,

which implies

$$\boxed{u_1' y_1' + u_2' y_2' = f(t)}$$

So, the coefficients u_1 & u_2 in

$$y_p = u_1 y_1 + u_2 y_2$$

satisfy

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(t) \end{cases}$$

(Note that the conditions are for u_1' & u_2'. We can solve the system to find u_1' & u_2', and then integrate to find u_1 & u_2.

Ex. Solve $y'' + y = \sec t$.

First note that we cannot solve this eqn. w/ method of undetermined coefficients ($\sec t$ is not one of admissible RHS)

(1) Solve $y'' + y = 0$

$r^2 + 1 = 0 \Rightarrow r^2 = -1 \Rightarrow r = \pm i$

$y_1 = \cos t \qquad y_2 = \sin t$

$y_c = c_1 \cos t + c_2 \sin t$

2 Look for $y_p = u_1 y_1 + u_2 y_2$

$$= u_1 \cos t + u_2 \sin t$$

System for u_1' & u_2' is

$$\begin{cases} u_1' \cos t + u_2' \sin t = 0 \\ -u_1' \sin t + u_2' \cos t = \sec t \end{cases}$$

Augmented matrix is

$$\begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & \sec t \end{bmatrix}$$

To set pivot 1, multiply row 1 by $\cos t$ and row 2 by $-\sin t$, and add (want to use $\cos^2 t + \sin^2 t = 1$)

$$\begin{bmatrix} 1 & 0 & -\dfrac{\sin t}{\cos t} \\ \cos t & \sin t & 0 \end{bmatrix}$$

$$\Rightarrow \quad u_1' = -\frac{\sin t}{\cos t} \quad \Rightarrow \quad u_1 = \ln|\cos t|$$

$$-\frac{\sin t}{\cos t} \cdot \cos t + \sin t \, u_2' = 0$$

$\Rightarrow u_2' = 1 \Rightarrow u_2 = t$

We do not need integration constants when we integrate u_1' & u_2' because we are trying to find one y_p, and not all possible ones.

We found $y_p = u_1 y_1 + u_2 y_2$

$= \ln|\cos t| \cos t + t \sin t$

and so the general solution is

$$y = \underbrace{c_1 \cos t + c_2 \sin t}_{y_c} + \left. \begin{array}{l} \ln|\cos t| \cos t \\ + t \sin t \end{array} \right\} y_p .$$

Ex. Solve

$$y'' + 2y' + y = e^{-t}$$

(1) Solve hom. eqn.

$$y'' + 2y' + y = 0$$

$$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0$$

$$\Rightarrow r = -1$$

$$y_c = c_1 e^{-t} + c_2 t e^{-t}$$

(2) $y_p = u_1 e^{-t} + u_2 t e^{-t}$

System for u_1' & u_2' is

$$\begin{cases} e^{-t} u_1' + t e^{-t} u_2' = 0 \\ (e^{-t})' u_1' + (t e^{-t})' u_2' = e^{-t} \end{cases}$$

$$\Leftrightarrow \begin{cases} u_1' + t u_2' = 0 \\ -\cancel{e^{-t}} u_1' + \cancel{e^{-t}} (-t+1) u_2' = \cancel{e^{-t}}\, 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} u_1' + t u_2' = 0 \\ -u_1' + (1-t) u_2' = 1 \end{cases}$$

Augmented matrix is

$$\begin{bmatrix} 1 & t & 0 \\ -1 & 1-t & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad u_2' = 1 \quad \Rightarrow \quad u_2 = t$$

and

$$u_1' + t u_2' = 0 \Leftrightarrow$$

$$u_1' = -t \Rightarrow u_1 = -t^2/2$$

Thus

$$y_p = -\frac{t^2}{2}e^{-t} + t \cdot te^{-t}$$
$$= \frac{t^2}{2}e^{-t}$$

and

$$y = \underbrace{c_1 e^{-t} + c_2 t e^{-t}}_{y_c} + \underbrace{\frac{t^2}{2}e^{-t}}_{y_p}$$

Note that we could have used method of undet. coeff. to solve. As an exercise, check that we would have found same solution with it.

Equations w/ variable coefficients

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = f(t)$$

Assume $a_2(t) \neq 0$ in some interval, so that we can divide by $a_2(t)$ and obtain eqn. in standard form

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$

We consider the IVP w/ initial cond.

$$y(t_0) = Y_0$$
$$y'(t_0) = Y_1$$

Thm. If p, q, g are continuous on some interval ~~(a,b)~~ (a,b) and $t_0 \in (a,b)$, then for any choice of Y_0 & Y_1 there exists a unique solution to the IVP $y(t)$, with $t \in (a,b)$.

Ex. $(1+t^2)y'' + ty' - y = \tan t$

$$y(1) = Y_0, \quad y'(1) = Y_1$$

Rewrite eqn. as

$$y'' + \frac{t}{1+t^2}\,y' - \frac{1}{1+t^2} = \frac{\tan t}{1+t^2}$$

In this case we have

$$p(t) = \frac{t}{1+t^2}, \quad q(t) = -\frac{1}{1+t^2}$$

$$g(t) = \frac{\tan t}{1+t^2}$$

p & q are continuous for all $t \in \mathbb{R}$

$g(t)$ continuous when $\cos t \neq 0$

Since $t_0 = 1$, we pick interval $(-\pi/2, \pi/2)$ on which all three p, q & g are cont.

For all values of y_0 & y_1, the IVP has unique sol. $y(t)$, $t \in (-\pi/2, \pi/2)$.

Ex. $t^2 y'' + t y' + y = \cos t$

$y(0) = 1, \quad y'(0) = 0$

Rewrite eqn. as

$$y'' + y'/t + y/t^2 = \frac{\cos t}{t^2}$$

~~p,q~~ $p = \frac{1}{t}, \quad q = 1/t^2, \quad g = \frac{\cos t}{t^2}$

Since p, q & g are not cont. at 0, but the initial conditions are prescribed at 0, the IVP is not well posed.

Def. An equation of the form

$$at^2y'' + bty' + cy = f(t)$$

called Cauchy - Euler's equation

Rmk. For these eqns we cannot assign initial cond. at 0.

We will always look for solutions to Cauchy - Euler's eqns in the form

$$y = t^r.$$

Ex. $t^2y'' + 7ty' - 7y = 0$

look for solutions $y = t^r$.

Compute derivatives and plug in eqn. to find r:

$y' = rt^{r-1}$, $y'' = r(r-1)t^{r-2}$

Plugging in gives

$$r(r-1)t^{r-2}t^2 + 7rt^{r-1}t - 7t^r = 0$$

$$\Leftrightarrow (r^2-r)t^r + 7rt^r - 7t^r = 0$$

Divide by t^r

$$r^2 - r + 7r - 7 = 0$$

$$\Leftrightarrow r^2 + 6r - 7 = 0 \quad \Leftrightarrow r_1 = 1, \; r_2 = -7$$

$$\Rightarrow y_1 = t \quad \& \quad y_2 = t^{-7}$$

General solution is

$$y = c_1 t + c_2 t^{-7}.$$

Ex. Solve IVP

$$t^2 y'' + 7ty' + 5y = 0$$

$$y(1) = -1, \quad y'(1) = 13$$

Look for solutions in form $y = t^r$, take derivatives, plug in:

$$y' = rt^{r-1} \qquad y'' = r(r-1)t^{r-2}$$

$$r(r-1)\cancel{t^r} + 7r\cancel{t^r} + 5\cancel{t^r} = 0$$

$$r^2 - r + 7r + 5 = 0$$

$r^2 + 6r + 5 = 0$

$r_1 = -5 \ , \ r_2 = -1$

general solution is

$$y = c_1 t^{-5} + c_2 t^{-1}$$

Apply init cond. $y(1) = -1$

$-1 = y(1) = c_1 + c_2 \Rightarrow c_1 + c_2 = -1$

To apply init. cond. $y'(1) = 13$

compute

$$y' = -5c_1 t^{-6} - c_2 t^{-2}$$

$13 = y'(1) = -5c_1 - c_2 \Rightarrow -5c_1 - c_2 = 13$

We have the system

$$\begin{cases} c_1 + c_2 = -1 \\ -5c_1 - c_2 = 13 \end{cases} \Rightarrow \quad -4c_1 = 12 \Rightarrow c_1 = -3$$

$$c_2 = -1 - c_1 = -1 + 3 = 2$$

$$y = -3t^{-5} + 2t^{-1}.$$

Ex. $(t+1)^2 y'' + 10(t+1)y' + 14y = 0$

This is a Cauchy-Euler eqn, w/ $(t+1)$ replacing t.

We look for solutions in the form (13)

$y = (t+1)^r$

$y' = r(t+1)^{r-1}$ $\quad y'' = r(r-1)(t+1)^{r-2}$

Substituting

$r(r-1)\cancel{(t+1)^r} + 10r\cancel{(t+1)^r} + 14\cancel{(t+1)^r} = 0$

$r^2 - r + 10r + 14 = 0$

$r^2 + 9r + 14 = 0$

$r = -7, -2 \longrightarrow y_1 = (t+1)^{-7}$

$y_2 = (t+1)^{-2}$

gen. sol. is

$y = c_1 (t+1)^{-7} + c_2 (t+1)^{-2}$.