Free Mechanical Oscillations

Want to take a closer look at homogeneous second order equations of the form

$$my'' + by' + ky = 0 \qquad (m, k > 0,\ b \geq 0)$$

In the spring-mass oscillator, m is the mass, b is the friction parameter (or damping coefficient), and k is the spring constant.

Case I: No damping ($b = 0$)

Rewrite eqn as

$$y'' + \frac{k}{m} y = 0$$

Let $\omega^2 = \frac{k}{m} \Rightarrow y'' + \omega^2 y = 0$

To solve,

$$r^2 + \omega^2 = 0 \Rightarrow r = \pm i\omega$$

$$y = c_1 \cos \omega t + c_2 \sin \omega t$$

From a physical point of view, it is more convenient to write solution in the form

$$y(t) = A \sin(\omega t + \phi)$$

Using the angle addition law for $\sin x$:

$$y(t) = A \cos \omega t \sin \phi + A \sin \omega t \cos \phi$$

But we already knew

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

Comparing the two expressions

$$c_1 = A \sin \phi$$

$$c_2 = A \cos \phi$$

We have

$$c_1^2 + c_2^2 = A^2 (\sin^2 \phi + \cos^2 \phi) = A^2$$

or

$$A = \sqrt{c_1^2 + c_2^2}$$

We call A the <u>amplitude</u> of the motion, since

$$|y| = |A \sin(\omega t + \phi)| = A |\sin(\omega t + \phi)| \leq A.$$

Also,

$$\tan \phi = \frac{\sin \phi}{\cos \phi} = c_1 / c_2$$

and so $\phi = \tan^{-1} \dfrac{c_1}{c_2}$

We call ϕ the <u>phase angle</u> of the motion. We call ω the <u>angular frequency</u>, and $T = \dfrac{2\pi}{\omega}$ is the <u>period</u>.

<u>EX</u>. $y'' + 64\, y = 0 \qquad y(0) = 1$
$\qquad\qquad\qquad\qquad\qquad\quad y'(0) = 0$

$\omega^2 = 64 \Rightarrow \omega = 8$

$y = c_1 \cos 8t + c_2 \sin 8t$

To apply initial cond., compute y'

$y' = -8\, c_1 \sin 8t + 8 c_2 \cos 8t$

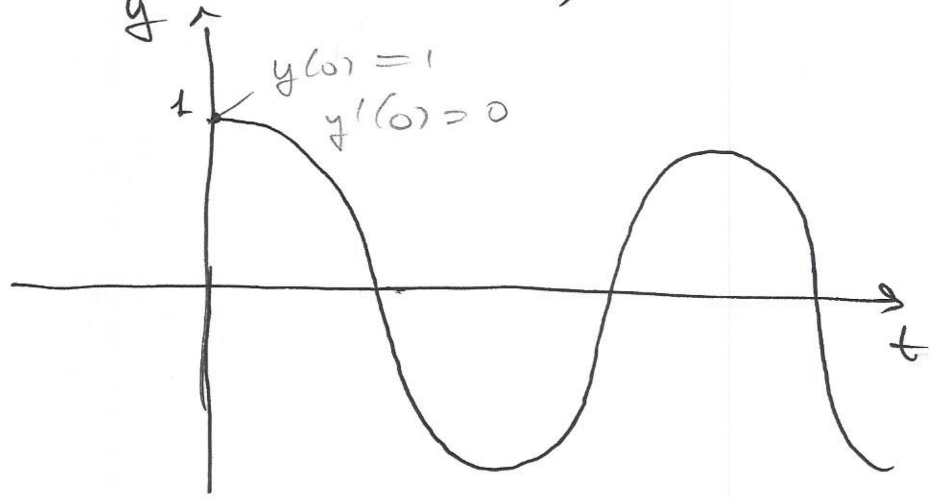
$$\left.\begin{array}{l} 1 = y(0) = c_1 \\ 0 = y'(0) = 8 c_2 \end{array}\right. \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = 0 \end{array}$$

$y = \cos 8t$

$A = \sqrt{c_1^2 + c_2^2} = 1$

$\phi = \tan^{-1} \left(c_1/c_2 \right) = \pi/2$

$y = \sin(\beta t + \pi/2)$



Case II : Damping

$$my'' + by' + ky = 0 \implies mr^2 + br + k = 0$$

$$\Rightarrow r = \frac{-b \pm \sqrt{b^2 - 4km}}{2m}$$

We distinguish three subcases, based on the sign of the discriminant $b^2 - 4km$.

Subcase 1. $\underline{b^2 - 4km < 0}$

Call $\alpha = -b/2m$, $\beta = \frac{\sqrt{4km - b^2}}{2m}$

Then the roots are

$$r = \alpha \pm i\beta$$

and the solution is

$$y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$$

$y = e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t)$

Proceeding as in Case I, let

$$A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \phi = \tan^{-1} c_1/c_2$$

Then

$$y(t) = Ae^{\alpha t} \sin(\beta t + \phi)$$

The damping factor is

$$Ae^{\alpha t} = Ae^{-b/2m\, t}$$

Since $b > 0$,

$$\lim_{t \to \infty} Ae^{-b/2m\, t} = 0$$

which means that the amplitude of the motion decreases over time, and eventually vanishes.

The quasi-period is $2\pi/\beta = \dfrac{4m\pi}{\sqrt{4mk - b^2}}$

and the quasi-frequency is $1/p$.

We call the system under-damped in this case, because the effect of damping is relatively small and still allows for oscillatory motion.

Ex. $y'' + 10y' + 64y = 0$

$y(0) = 1 \quad , \quad y'(0) = 0$

Solve $\quad y'' + 10y' + 64y = 0$

$$r^2 + 10r + 64 = 0$$

$$r = -5 \pm \sqrt{39}\, i$$

$\alpha = -5 \qquad \beta = \sqrt{39}$

$$y = e^{-5t}\left(c_1 \cos\sqrt{39}\,t + c_2 \sin\sqrt{39}\,t\right)$$

$$y' = e^{-5t}\left(-5c_1 \cos\sqrt{39}\,t - \sqrt{39}\,c_1 \sin\sqrt{39}\,t - 5c_2 \sin\sqrt{39}\,t + \sqrt{39}\,c_2 \cos\sqrt{39}\,t\right)$$

Apply initial cond.

$1 = y(0) = c_1 \quad \Rightarrow \quad c_1 = 1$

$0 = y'(0) = -5c_1 + \sqrt{39}\,c_2 \Rightarrow c_2 = \dfrac{5\sqrt{39}}{39}$

$$A = \sqrt{c_1^2 + c_2^2} = \sqrt{1 + \frac{25}{39}} = \sqrt{\frac{64}{39}} = \frac{8\sqrt{39}}{39}$$

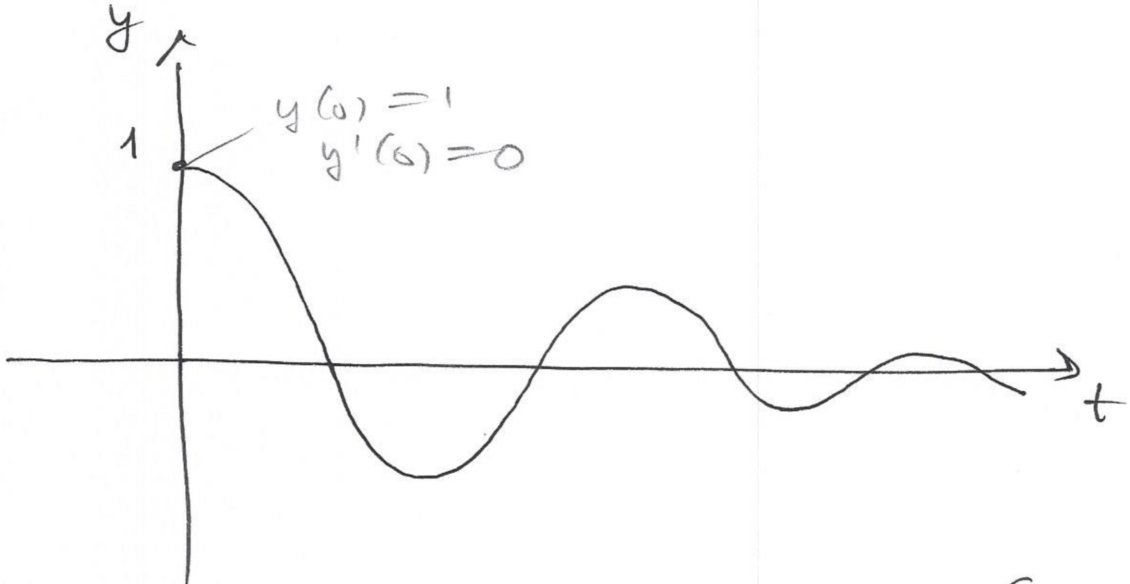
$$\phi = \tan^{-1}\frac{\sqrt{39}}{5}.$$

$$P = 2\pi/\sqrt{39} = \frac{2\pi\sqrt{39}}{39}$$

Quasi frequency $\quad \dfrac{\sqrt{39}}{2\pi}.$

Solution can be written as

$$y = \frac{8\sqrt{39}}{39} e^{-5t} \sin\left(\sqrt{39}\,t + \tan^{-1}\frac{\sqrt{39}}{5}\right).$$



Subcase 2: $b^2 - 4km = 0$ (critically damped)

Then roots are real & coincident

$$r_{1,2} = -b/2m$$

and the general solution is

$$y = e^{-b/2m}(c_1 + c_2 t)$$

Ex. $y'' + 16y' + 64y = 0$ $\quad y(0) = 1$, $y'(0) = 0$

Solve $y'' + 16y' + 64y = 0$

$$r^2 + 16r + 64 = 0 \qquad (r+8)^2 = 0$$

$$r = -8$$

$$y = e^{-8t}(c_1 + c_2 t)$$

$$y' = e^{-8t}(-8c_1 - 8c_2 t + c_2)$$

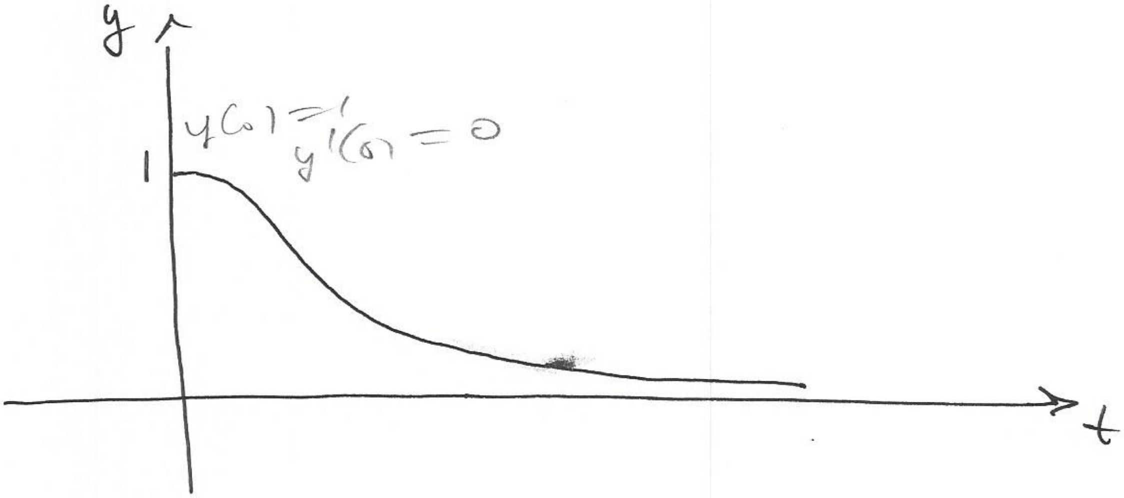
Apply initial cond.

$$1 = y(0) = c_1 \Rightarrow c_1 = 1$$

$$0 = y'(0) = -8c_1 + c_2 \Rightarrow c_2 = 8$$

Solution is

$$y = e^{-8t}(1 + 8t)$$



Note that $y > 0$ for all $t \geq 0$

$$\lim_{t \to \infty} y(t) = 0$$

Subcase 3 : $b^2 - 4km > 0$ (Overdamped)

Two real roots r_1, r_2

and solution is

$$y = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$$

Ex. $y'' + 20y' + 64y = 0$ $\quad y(0) = 1$, $y'(0) = 0$

Solve eqn.

$$r^2 + 20r + 64 = 0 \qquad r = -4, -16$$

and gen. sol. is

$$y = c_1 e^{-4t} + c_2 e^{-16t}$$

Compute

$$y' = -4c_1 e^{-4t} - 16 c_2 e^{-16t}$$

Apply initial cond.

$$\begin{cases} 1 = y(0) = c_1 + c_2 \\ 0 = y'(0) = -4c_1 - 16c_2 \end{cases}$$

$$c_1 = -4c_2$$

$$-4c_2 + c_2 = 1 \quad \rightarrow \quad c_2 = -1/3, \quad c_1 = 4/3$$

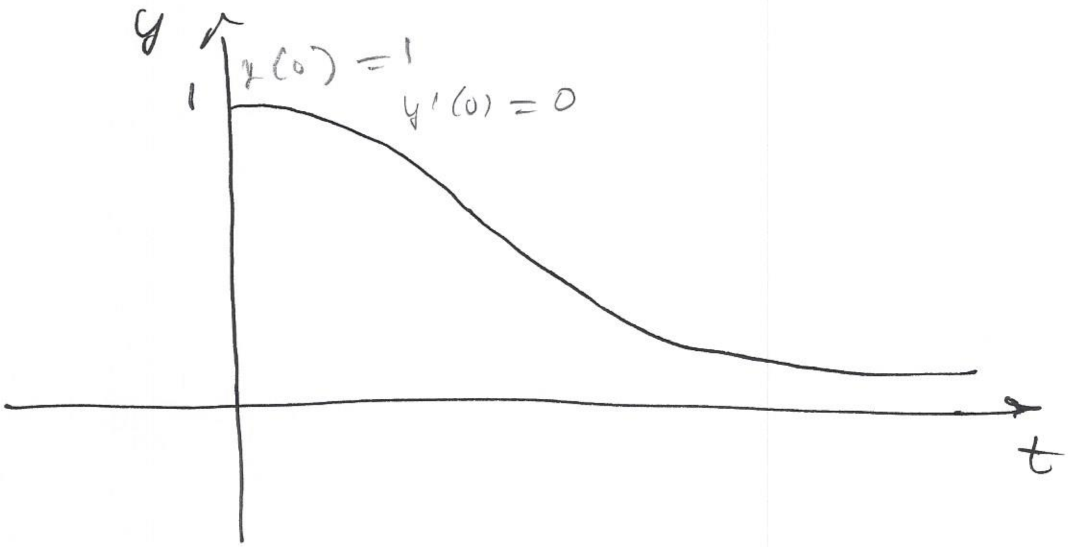
$$\Rightarrow \quad y = \tfrac{4}{3} e^{-4t} - \tfrac{1}{3} e^{-16t}$$

$$= \tfrac{1}{3} e^{-4t} (4 - e^{-12t})$$

Note that

$y \geq 0$ for all $t \geq 0$

$\lim_{t \to \infty} y(t) = 0$



Remarks ① The two subcases critically damped and overdamped are very similar: the effect of damping is very strong, and this leads to no oscillatory motion. The motion will get close to zero very fast.

② In the cases critically damped and overdamped, it is possible for the solution to cross the t-axis. It depends on the initial conditions.

(3) To draw the graph of the motion, pay attention to initial conditions.
The condition $y(0) = Y_0$ tells you where the graph intersects the y-axis.
The condition $y'(0) = Y_1$ tells you if the graph goes up from initial condition (when $y_1 > 0$), if it has a maximum at $t = 0$ ($Y_1 = 0$), or it goes down ($y_1 < 0$).
Look for possible t-intercepts, and also check for a possible minimum by setting $y'(t) = 0$.