10/18/19 — Linear differential equations of order n —

Write $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, ... These are linear transformations on function spaces $D: C^1 \to C^0$, $D^k: C^k \to C^0$. More generally, a linear operator of order n is $L = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$, and $L: C^n \to C^0$. Assume a_i continuous functions.

Linear differential equations of order n, $Ly = F$. There exists a unique solution given initial conditions $y(0) = y_0$, $y'(0) = y_1, \dots, y^{(n-1)}(0) = y_{n-1}$. If $F = 0$, $\mathrm{Ker}(L)$ is an n-dimensional vector space, i.e. $y_h = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ is the general solution. If $F \neq 0$, the general solution is $y = y_p + y_h$. Note y_p (particular) and y_h (homogeneous).

p. 459 / 32, 35, 36

<u>Theorem</u> Let $a_1, a_2, \dots, a_n$, and F be functions that are continuous on an interval I. Then for any $x \in I$, the initial value problem $Ly = F(x)$, $y(x_0) = y_0$, $y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$ has a unique solution on the interval I.

<u>Problem 1</u> Determine two linearly independent solutions $2x^2 y'' + 5xy' + y = 0$ $\quad x > 0$

<u>Problem 2</u> Determine a particular solution of $y'' + y' - 6y = 18e^{3x}$

<u>Problem 3</u> Determine a particular solution of $y'' + y' - 2y = 4x^2$

10/21/19 — Constant coefficient homogeneous linear equations — Given $y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$. Trial solution $y = e^{rx}$. Reduce to finding zeros of the polynomial $r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$. Need refinements for multiple roots and complex roots. Use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

Do p. 468 / 1, 3, 6

<u>Remark</u> In principle, auxiliary polynomial always factors over the complex numbers as $(r-r_1)^{m_1}(r-r_2)^{m_2}\dots(r-r_k)^{m_k}$. This is the fundamental theorem of algebra. This is an existence statement, not an algorithm to find the roots. Quadratic formula has manageable extension to cubics and quartics.

<u>Remark</u> Double root method explained by limiting argument

$$\lim_{r_1 \to r_2} \frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2} = \frac{d}{dr}\left(e^{rt}\right)\Big|_{r=r_2} = te^{r_2 t}$$

Euler's formula justified by power series

$$e^{i\theta} = \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} = \sum_{\substack{k=2r \\ r=0}}^{\infty} (-1)^r \frac{\theta^{2r}}{(2r)!} + i \sum_{\substack{k=2r+1 \\ r=0}}^{\infty} (-1)^r \frac{\theta^{2r+1}}{(2r+1)!}$$

so

$$e^{i\theta} = \cos\theta + i\sin\theta$$

<u>Problem 4</u> Determine a basis for the solution space $y'' + 2y' - 3y = 0$

<u>Problem 5</u> Determine a basis for the solution space $y'' - 6y' + 25y = 0$

<u>Problem 6</u> Find the general solution of $y'' - 6y' + 9y = 0$.

10/23/19 — Constant coefficient homogeneous linear equations (Continued)

Recall notation $L = D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n$. Factor $L = (D-r_1)^{m_1}(D-r_2)^{m_2}\cdots(D-r_k)^{m_k}$. Then for each real root get m_i solutions to $Ly=0$, i.e. $e^{r_i x}, xe^{r_i x}, \dots, x^{m_i-1}e^{r_i x}$. For complex roots $r_i = s_i \pm \sqrt{-1}\, t_i$, get $2m_i$ solutions by taking real and imaginary parts $e^{s_i x}\cos t_i x$ and $e^{s_i x}\sin t_i x$ times powers of x, $1, x, x^2, \dots, x^{m_i-1}$.

Do p. 468/21, 31, [illegible]-34, 39

__Remark__ Conjugation gives $e^{-rx}(D-r)(e^{rx}f) = Df$ and by iteration $e^{-rx}(D-r)^{\ell}(e^{rx}f) = D^{\ell}f$. Reduce to $r=0$, gives reason for prescription above. Point is that $1, x, x^2, \dots, x^{\ell-1}$ are obviously in the kernel of D^{ℓ}. So the corresponding products $x^q e^{rx}$, $1 \le q \le \ell-1$, are in kernel of $(D-r)^{\ell}$.

__Problem 7__ Determine a basis for the solution space $(D^2+4)^2(D+1)y=0$

__Problem 8__ Determine a basis for the solution space $(D^2+9)^3 y=0$

__Problem 9__ Solve $y'''-y''+y'-y=0$, $y(0)=0$, $y'(0)=1$, $y''(0)=2$

__Problem 10__ See notes of 4/10/20.

4/10/20

Problem involving Laplaces equation - partial differential equations
$u_{xx} + u_{yy} = 0$, p. 469, # 39, section 6.2.

(a) Substitute $u(x,y) = e^{x/\alpha} f(\xi)$, where $\xi = \beta x - \alpha y$, $\alpha > 0$ and $\beta > 0$ into Laplaces equation to obtain a linear differential equation

Calculate $u_x = \left(\frac{1}{\alpha}\right) e^{x/\alpha} f(\beta x - \alpha y) + \beta e^{x/\alpha} f'(\beta x - \alpha y)$ and
$u_{xx} = \left(\frac{1}{\alpha}\right)^2 e^{x/\alpha} f(\beta x - \alpha y) + \frac{2\beta}{\alpha} e^{x/\alpha} f'(\beta x - \alpha y) + \beta^2 e^{x/\alpha} f''(\beta x - \alpha y)$

Moreover $u_y = -\alpha e^{x/\alpha} f'(\beta x - \alpha y)$, $u_{yy} = \alpha^2 e^{x/\alpha} f''(\beta x - \alpha y)$,
so $\Delta u = u_{xx} + u_{yy} = (\alpha^2 + \beta^2) e^{x/\alpha} f''(\beta x - \alpha y) + \frac{2\beta}{\alpha} e^{x/\alpha} f'(\beta x - \alpha y) + \left(\frac{1}{\alpha}\right)^2 e^{x/\alpha} f(\beta x - \alpha y)$.

Thus $\Delta u = 0$ reduces to $(\alpha^2 + \beta^2) f''(\xi) + \frac{2\beta}{\alpha} f'(\xi) + \frac{1}{\alpha^2} f(\xi) = 0$.
Normalizing the lead term given $f''(\xi) + 2p f'(\xi) + \frac{q}{\alpha^2} f(\xi) = 0$,
where $p = \frac{\beta}{\alpha(\alpha^2+\beta^2)}$, $\frac{q}{\alpha^2} = \frac{1}{\alpha^2(\alpha^2+\beta^2)}$.

(b) Solve the second order linear homogeneous equation from (a),
$f''(\xi) + \frac{2\beta}{\alpha} f'(\xi) + \frac{1}{\alpha^2} f(\xi) = 0$,

$e^{k\xi}$ Substitute $e^{k\xi}$ giving $k^2 + 2pk + \frac{q}{\alpha^2} = 0$, and by the quadratic formula $k = -p \pm \sqrt{p^2 - q/\alpha^2} = -p \pm \frac{1}{\alpha}\sqrt{\alpha^2 p^2 - q}$

We proceed to rewrite the expression for k. Note that $\alpha^2 p^2 - q = \beta^2(\alpha^2+\beta^2)^{-2} - (\alpha^2+\beta^2)^{-1} = -\alpha^2(\alpha^2+\beta^2)^{-2}$. So $\sqrt{\alpha^2 p^2 - q} = \pm \alpha i (\alpha^2+\beta^2)^{-1}$ and thus $k = \frac{-\beta}{\alpha(\alpha^2+\beta^2)} \pm \frac{i}{(\alpha^2+\beta^2)}$
and $k = \left(-\frac{\beta}{\alpha} \pm i\right)(\alpha^2+\beta^2)^{-1}$.

Returning to the differential equation, we choose the root $k = (\alpha^2+\beta^2)^{-1}\left(-\frac{\beta}{\alpha} + i\right)$ so that $k\xi = (-p + iq)\xi$.

Thus $f(\xi) = e^{k\xi} = e^{-p\xi} e^{iq\xi} = e^{-p\xi}(\cos q\xi + i \sin q\xi)$.
The general solution of the homogeneous differential equation (b) is $e^{-p\xi}(c_1 \cos(q\xi) + c_2 \sin(q\xi))$.

Finally returning to the partial differential equation (a) we obtain

$$u(x,y) = e^{x/\alpha} e^{-p\xi}(c_1 \cos q\xi + c_2 \sin q\xi)$$

4/14/20

Problem 1. Determine two linearly independent solutions of $2x^2y'' + 5xy' + y = 0$, $x > 0$

$\boxed{y = x^r}$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$

$2x^2 r(r-1)x^{r-2} + 5xrx^{r-1} + x^r = 0$

$(2r(r-1) + 5r + 1)x^r = 0$

$2r^2 + 3r + 1 = 0$, $(2r+1)(r+1) = 0$, $r = -1, -1/2$

$y = x^{-1}, x^{-1/2}$ $\qquad y_h = c_1x^{-1} + c_2x^{-1/2}$

Problem 2 Determine a particular solution to

$y'' + y' - 6y = 18e^{3x}$. Try $Ae^{3x} = y$,

$y' = 3Ae^{3x}$, $y'' = 9Ae^{3x}$

$9Ae^{3x} + 3Ae^{3x} - 6Ae^{3x} = 18e^{3x}$

$6Ae^{3x} = 18e^{3x}$, $6A = 18$, $A = 3$ $\quad \boxed{y = 3e^{3x}}$

Problem 4 $y'' + 2y' - 3y = 0$ $\boxed{y = e^{rx}}$

$r^2 + 2r - 3 = 0$ $\quad (r+3)(r-1) = 0$

$r = 1, -3$, $y_h = c_1e^x + c_2e^{-3x}$

Problem 5 $y'' - 6y' + 25y = 0$, $\boxed{y = e^{rx}}$

$r^2 - 6r + 25 = 0$

$(r-3)^2 + 16 = 0$, $(r-3)^2 = -16$, $r = 3 + 4i$ is a root

$e^{rx} = e^{3x}e^{4xi} = e^{3x}(\cos 4x + i\sin 4x)$

$y_h = c_1e^{3x}\cos 4x + c_2e^{3x}\sin 4x$

Problem 6 $y'' - 6y' + 9y = 0 \quad y = e^{rx}$

$r^2 - 6r + 9 = 0, \ (r-3)^2 = 0, \ r = 3$

$y = c_1 e^{3x} + c_2 x e^{3x} \qquad y = e^{3x}$

Problem 7 $(D^2+4)^2 (D+1) y = 0$

$D^2 + 4 \quad \cos 2x, \sin 2x, x\cos 2x, x \sin 2x$

$D + 1 \quad e^{-x} \qquad y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin x$

$+ c_5 e^{-x}$

Problem 8 $(D^2+9)^3 y = 0$

$D^2 + 9 \quad \cos 3x, \sin 3x, x\cos 3x, x \sin 3x, x^2 \cos 3x, x^2 \sin 3x$

$y = c_1 \cos 3x + c_2 \sin 3x + c_3 x \cos 3x + c_4 x \sin 3x + c_5 x^2 \cos 3x + c_6 x^2 \sin 3x$

Problem 9 $y''' - y'' + y' - y = 0, \ \boxed{y(0) = 0, y'(0) = 1, y''(0) = 2}$

$y = e^{rx} \quad r^3 - r^2 + r - 1 = 0$

$r^2(r-1) + r - 1 = (r^2+1)(r-1) = 0 \quad r = 1, \ r = \pm i$

$\boxed{y = c_1 e^x + c_2 \cos x + c_3 \sin x} \qquad y(0) = c_1 + c_2 = 0$

$y' = c_1 e^x - c_2 \sin x + c_3 \cos x \qquad y'(0) = c_1 + c_3 = 1$

$y'' = c_1 e^x - c_2 \cos x - c_3 \sin x \qquad y''(0) = c_1 - c_2 = 2$

$(c_1 + c_2) + (c_1 - c_2) = 0 + 2, \ 2c_1 = 2, \ c_1 = 1$

$c_1 + c_3 = 1, \ 1 + c_3 = 1, \ c_3 = 0$

$c_1 + c_2 = 0, \ 1 + c_2 = 0, \ c_2 = -1 \qquad \boxed{y = e^x - \cos x}$

Problem 3 $y'' + y' - 2y = 4x^2$. Find a particular solution

$$\boxed{y = Ax^2 + Bx + C}$$

$$y' = 2Ax + B$$

$$y'' = 2A$$

$$2A + 2Ax + B - 2Ax^2 - 2Bx - 2C = 4x^2$$

$-2A = 4$, $2A - 2B = 0$, $2A + B - 2C = 0$

$A = -2$, $B = A = -2$, $-4 - 2 = 2C$

$2C = -6$, $C = -3$

$$\boxed{y = -2x^2 - 2x - 3}$$

$\boxed{\text{Check}}$ $y' = -4x - 2$, $y'' = -4$

$$y'' + y' - 2y = -4 - 4x - 2 + 4x^2 + 4x + 6 = 4x^2$$