10/25/19 — Undetermined coefficients —

Consider inhomogeneous problem $y'' + ay' + by = F(x)$, where a and b are constant. Suppose $F(x)$ is a polynomial, exponential, cosine or sine or product of these. Try solution of similar type for y_p.

Do p. 481/18, 22, 25, 27 and p. 481/23 ~~[illegible]~~

Combine $y = y_p + y_h$ to obtain general solution.

<u>Problem 1</u> Solve $y'' + y = 6e^{x}$, particular solution

<u>Problem 2</u> Solve $y'' + 2y' + 5y = 3\sin 2x$, particular solution

✓ <u>Problem 3</u> Solve $y''' + 2y'' - 5y' - 6y = 4x^2$; general solution

✓ <u>Problem 4</u> Solve $y''' + 3y'' + 3y' + y = 2e^{-x} + 3e^{2x}$, particular solution

<u>Remark</u> The same idea works in principle for linear equations of order $n \geq 2$. Of course, the calculations may become cumbersome for very large n.

10/28/19 — Annihilators —

If $P(D)\,y = F(x)$ and $Q(D)\,F = 0$, then $P(D)\,Q(D)\,y = 0$. Explain test solutions for undetermined coefficients. Here $D = \frac{d}{dx}$ and P, Q are polynomials with constant coefficients.

Do p. 481/18, 22, ~~[illegible]~~, 25, 27, 23

Note that $y = y_h + y_p$ where y_h is the general solution to the homogeneous equation $P(D)y = 0$.

If extra time, introduce method of complex valued trial solutions.

Do p. 484/1, 5

$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

Problem 5 Solve $(D^2 + 6)\,y = \sin^2 x \cos^2 x$ (use double angle formulas)

Problem 6 Solve $y'' + 2y' + y = 50 \sin 3x$ (complexify)

✓ Problem 7 Solve $y'' - 4y = 100\, x e^x \sin x$ (complexify $e^{(1+i)x}$) (two methods)

Remark The annihilator method is perhaps most valuable when $F(x)$ solves the homogeneous problem, i.e. $P(D)\,F(x) = 0$. In that situation, it is somewhat more difficult to guess the right form of the trial solution.

11/1/19 — Variation of parameters — Higher order —

Given $y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_n(x) y = F(x)$. Suppose $c_1 y_1 + c_2 y_2 + \dots + c_n y_n$ solves homogeneous. Try test solution $u_1 y_1 + u_2 y_2 + \dots + u_n y_n$. Need

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & & & \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ F \end{pmatrix}$$

Solve for $u_1', u_2', \dots, u_n'$ by Cramer's rule or row reduction and integrate to get $u_1, u_2, \dots, u_n$.

Do p. 512/20. First by $y = e^x z$, $(D-1)y = e^x Dz$, $(D-1)^k y = e^x D^k z$, by induction on k. Second by variation of parameters.

Do p. 513/31 by direct calculation.

✓ <u>Problem 8</u> Solve $y''' - 3y'' + 3y' - y = 2x^{-2} e^x$ (two methods)

✓ <u>Problem 9</u> Show that $y_p = \frac{1}{2} \int_a^x F(t)(x-t)^2 e^{r(x-t)}\, dt$ solves $(D-r)^3 y = F(x)$

4/16/20

Variation of parameters – Higher Order

Problem 1. $y''' - 3y'' + 3y' - y = 2x^{-2}e^{x}$, $x > 0$
Rewrite as $(D-1)^3 y = 2x^{-2}e^{x}$
Change dependent variable $y = e^{x} z$, $(D-1)y = e^{x} Dz$, $(D-1)^2 y = e^{x} D^2 z$, $(D-1)^3 y = e^{x} D^3 z$.
So $D^3 z = 2x^{-2}$, $D^2 z = -2x^{-1}$, $Dz = -2\ln x$
$z = -2(x\ln x - x)$. So $y_p = -2(x \ln x - x)e^{x}$
and $y_h = c_1 e^{x} + c_2 x e^{x} + c_3 x^{-2} e^{x}$

Problem 2. Show that a particular solution to $(D-r)^3 y = F(x)$ is $y_p = \frac{1}{2}\int_a^x F(t)(x-t)^2 e^{r(x-t)}\,dt$
Differentiate three times, noting the two places where x appears: $(D-r)y_p = \frac{1}{2}\int_a^x F(t)\left((D-r)(x-t)^2 e^{r(x-t)}\right)dt$
$= \int_a^x F(t)(x-t)e^{r(x-t)}\,dt$, $(D-r)^2 y_p = \int_a^x F(t)e^{r(x-t)}\,dt$,
and finally, $(D-r)^3 y_p = F(x)$.

Method of Undetermined Coefficients – Annihilators

Problem 1. $y''' + 3y'' + 3y' + y = 2e^{-x} + 3e^{2x}$. Rewrite as $(D+1)^3 y = 2e^{-x} + 3e^{2x}$. Apply $(D+1)(D-2)$ to annihilate right hand side. So $(D+1)^4(D-2)y = 0$. So use $y_p = Ae^{2x} + Bx^3 e^{-x}$, $(D+1)^3 y = 27Ae^{2x} + 6Be^{-x}$.
So $y_p = \frac{1}{9}e^{2x} + \frac{1}{3}x^3 e^{-x}$. Also $y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$.
The general solution is $y_p + y_h$.

Problem 2 $y''' + 2y'' - 5y' - 6y = 4x^2$

(a) Homogeneous problem $y = e^{rx}$, $r^3 + 2r^2 - 5r - 6 = 0$

Factor $r^3 + 2r^2 - 5r - 6 = (r+1)(r^2 + r - 6) = (r+1)(r+3)(r-2)$.

So $r = -1, -3, 2$, and $y_h = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{2x}$

(b) Particular solution $y = Ax^2 + Bx + C$, $y' = 2Ax + B$,

$y'' = 2A$, $y''' = 0$. Substitute into original equation.

$4A - 10Ax - 5B - 6Ax^2 - 6Bx - 6C = 4x^2$, and

$-6Ax^2 - 10Ax - 6Bx + 4A - 5B - 6C = 4x^2$.

Equate coefficients $-6A = 4$, $-10A - 6B = 0$,

$4A - 5B - 6C = 0$. Thus $A = -\frac{2}{3}$ $B = -\frac{5}{3}A = \frac{10}{9}$

$C = \frac{1}{6}(4A - 5B) = \frac{2}{3}A - \frac{5}{6}B =$

$C = -\frac{4}{9} - \frac{15}{27} = -\frac{37}{27}$. So $y_p = -\frac{2}{3}x^2 + \frac{10}{9}x - \frac{37}{27}$.

$y_p = -\frac{2}{3}x^2 + \frac{10}{9}x - \frac{37}{27}$

(c) General solution, $y = y_p + y_h = -\frac{2}{3}x^2 + \frac{10}{9}x - \frac{37}{27} + c_1 e^{-x}$
$+ c_2 e^{-3x} + c_3 e^{2x}$.

Problem 3 $y'' - 4y = 100\, x e^x \sin x$. Trial solution given

by $y = (Ax + B)e^x \cos x + (Cx + D)e^x \sin x$. Differentiate

$y' = Ae^x \cos x + Ce^x \sin x + (Ax + B)e^x(\cos x - \sin x) +$

$(Cx + D)e^x(\cos x + \sin x) = (A + B + D)e^x \cos x$

$+ (C - B + D)e^x \sin x + (A + C)xe^x \cos x + (C - A)e^x \sin x$

Becomes complicated. Try alternative method.

Problem 3 $y'' - 4y = 100 x e^{x} \sin x = 100 x \, \mathrm{Im}(e^{(1+i)x})$.

Complexify $y'' - 4y = 100 x e^{(1+i)x}$. Trial solution

$y = (Ax + B) e^{(1+i)x}$, $y' = Ae^{(1+i)x} + (Ax+B)(1+i)e^{(1+i)x}$

$y' = (A + B(1+i)) e^{(1+i)x} + Ax(1+i) e^{(1+i)x}$

$y'' = (A + B(1+i))(1+i) e^{(1+i)x} + A e^{(1+i)x}(1+i)$

$+ 2iAx e^{(1+i)x} = [(2+2i)A + 2iB] e^{(1+i)x}$

$+ 2iAx e^{(1+i)x}$

$y'' - 4y = [(2+2i)A + 2iB - 4B] e^{(1+i)x} + [2iA - 4A]$

$x e^{(1+i)x} = 100 x e^{(1+i)x}$

$$A = \frac{100}{2i-4} = \frac{50}{-2+i} = \frac{50(-2-i)}{5} = 10(-2-i)$$

$$B = \frac{(2+2i)A}{4-2i} = \frac{(1+i)A}{2-i} = \frac{(1+i)(2+i)A}{5} = -2(1+i)(2+i)^2$$

$$= (2-14i)$$

$y = [(-20-10i)x + (2-14i)] e^{x} e^{ix}$

$= [(-20-10i)x + (2-14i)](\cos x + i \sin x) e^{x}$

$y_p = \mathrm{Im}\, y = (-10x\cos x - 20 x \sin x - 14\cos x + 2\sin x) e^{x}$

General solution is $y_p + y_h$, where $y_h = c_1 e^{2x} + c_2 e^{-2x}$

Problem 3 (alternative solution) $y'' - 4y = 100 x e^{x} \sin x$. Use undetermined coefficients to find particular solution

Try $y = Axe^{x}\cos x + Bxe^{x}\sin x + Ce^{x}\cos x + De^{x}\sin x$

Calculate first derivative

$y' = (A+B)xe^{x}\cos x + (-A+B)xe^{x}\sin x +$

$(A + C + D)e^{x}\cos x + (B - C + D)e^{x}\sin x$

Calculate $y'' = 2Bxe^x\cos x - 2Axe^x\sin x + (2A+2B+2D)e^x\cos x + (-2A+2B-2C)e^x\sin x$

Equation $y'' - 4y = 100xe^x\sin x$ to get system of equations $2B - 4A = 0$, $-2A - 4B = 100$, $2A + 2B - 4C + 2D = 0$, $-2A + 2B - 2C - 4D = 0$. Consequently, solve for A, B, C, D giving $y_p = -10xe^x\cos x - 20xe^x\sin x - 14e^x\cos x + 2e^x\sin$

Problem 1 on (Variation of parameters) – Alternative solution

$y''' - 3y'' + 3y' - y = 2x^{-2}e^x$. Note $y_h = c_1e^x + c_2xe^x + c_3x^2e^x$.

$$\begin{pmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x + e^x & 2xe^x + x^2e^x \\ e^x & xe^x + 2e^x & 2e^x + 4xe^x + x^2e^x \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x^{-2}e^x \end{pmatrix}$$

Now form augmented matrix and row reduce

$$\begin{pmatrix} 1 & x & x^2 & 0 \\ 1 & x+1 & x^2+2x & 0 \\ 1 & x+2 & x^2+4x+2 & 2x^{-2} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & x & x^2 & 0 \\ 0 & 1 & 2x & 0 \\ 0 & 0 & 1 & x^{-2} \end{pmatrix}$$

So $u_3' = x^{-2}$ and $u_3 = -x^{-1}$. Moreover $u_2' = -2xu_3' = -2x^{-1}$, giving $u_2 = -2\ln x$. Finally $u_1' = -xu_2' - x^2u_3' = 1$ yielding $u_1 = x$.

The particular solution is $u_1e^x + u_2xe^x + u_3x^2e^x$, or equivalently $y_p = -2xe^x\ln x$. Note that solutions of the homogeneous problem can be added to a particular solution giving another particular solution. In particular, y_p is not unique.