HOMEWORK #11 - MA 504

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Chapter 6, problem 1. Suppose α increases on $[a,b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.

Solution.
Let $P = \{y_0, y_1, ..., y_n\}$ be a partition of $[a,b]$, ie

$$a = y_0 \leq y_1 \leq \cdots \leq y_{n-1} \leq y_n = b.$$

We have that there exist y_j, $j = 0, ..., n$, such that $x_0 = y_j$ or $y_{j-1} < x < y_j$. In either case, we see that

$$U(P, f, \alpha) \leq \alpha(y_j) - \alpha(y_{j-1}) + \alpha(y_{j+1}) - \alpha(y_j) = \alpha(y_{j+1}) - \alpha(y_{j-1}),$$
$$L(P, f, \alpha) = 0.$$

Since α is continuous at x_0 we see that

$$|\alpha(y_{j+1}) - \alpha(y_{j-1})| \to 0 \quad \text{as } i \to \infty.$$

So

$$\inf U(P, f, \alpha) = 0.$$

Therefore $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.

Chapter 6, problem 2. Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\displaystyle\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

Solution.
Suppose that there exists a x_0 such that $f(x_0) > 0$. Since f is continuous there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < \frac{f(x_0)}{2}, \quad |x - x_0| < \delta.$$

So

$$|f(x)| \geq |f(x_0)| - |f(x) - f(x_0)| > f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2}.$$

Hence, since $f \geq 0$,

$$\int_a^b f(x)dx \geq \int_{x_0-\delta}^{x_0+\delta} f(x)dx > \frac{f(x_0)}{2}[x_0 + \delta - (x_0 - \delta)] = \delta f(x_0) > 0,$$

a contradiction. Therefore $f(x) = 0$ for all $x \in [a,b]$.