HOMEWORK #12 - MA 504

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Chapter 6, problem 4. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Solution.
Let $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$ be a partition of $[a, b]$, call it P. We have, by the density of the rationals (respectively the irrationals) on $\mathbb{R}$,

$$1 = M_i = \sup f(x) \quad (x_{i-1} \leq x \leq x_i),$$

$$0 = m_i = \inf f(x) \quad (x_{i-1} \leq x \leq x_i),$$

so

$$U(P, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = b - a,$$

$$L(P, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = 0.$$

Since P is an arbitrary partition of $[a, b]$, we have

$$\overline{\int_a^b} f dx = b - a > 0 = \underline{\int_a^b} f(x) dx.$$

Hence $f \notin \mathcal{R}$ on $[a, b], a < b$.
Chapter 6, problem 5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$ on $[a, b]$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?
Solution.
Answer to the first question: NÃO! (NO!).
Indeed, let $f(x) = 1$ for all irrational x, $f(x) = -1$ for all rational x. Similarly as we showed in the previous problem, one can show that

$$\overline{\int_a^b} f dx = b - a > 0 > a - b = \underline{\int_a^b} f(x) dx.$$

Hence $f \notin R$, but $f^2(x) = 1$ for all x, so $f^2 \in \mathcal{R}$ on $[a, b]$.
Answer to the second question: SIM! (YES!).
We have that $\phi(x) = \sqrt[3]{x}$ is continuous on $[a, b]$ for any $a, b \in \mathbb{R}$, so by theorem 6.11 if $f^3 \in \mathcal{R}$, then $h = \phi \circ f^3 \in \mathcal{R}$, where $h(x) = \phi(f^3(x)) = f(x)$, ie, $h = f$.

Chapter 6, problem 10. Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.
(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.
(b) If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p \, \mathrm{d}\alpha = 1 = \int_a^b g^q \, \mathrm{d}\alpha,$$

then

$$\int_a^b fg \, \mathrm{d}\alpha \leq 1.$$

(c) If f and g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left|\int_a^b fg \, \mathrm{d}\alpha\right| \leq \left\{\int_a^b |f|^p \, \mathrm{d}\alpha\right\}^{1/p} \left\{\int_a^b |g|^q \, \mathrm{d}\alpha\right\}^{1/q}.$$

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 7 and 8.

Solution.
First of all, assume $u, v > 0$, otherwise the inequality is trivial. We see that by making the substitution $\tilde{u} = u^p > 0$ and $\tilde{v} = v^q > 0$ the inequality that we want to show is equivalent to the following inequality

$$\left(\frac{\tilde{u}}{\tilde{v}}\right)^{1/p} \leq \frac{1}{p}\left(\frac{\tilde{u}}{\tilde{v}}\right) + \frac{1}{q}.$$

Now if we make the substitution $z = \dfrac{\tilde{u}}{\tilde{v}}$ and assume without loss of generality $\tilde{u} \geq \tilde{v}$, so $z \geq 1$, it suffices to show

$$z^{1/p} \leq \frac{z}{p} + \frac{1}{q},$$

whenever $z \geq 1$, and equality holds if and only if $z = 1$.
Now the previous inequality is equivalent, by making $x = z^{1/p} \geq 1$ to

$$0 \leq \frac{x^p}{p} - x + \frac{1}{q}.$$

Let $f(x) = \dfrac{x^p}{p} - x + \dfrac{1}{q}$. We have $f(1) = 0$ and

$$f'(x) = x^{p-1} - 1 > 0,$$

whenever $x > 1$. So f is a stricly increasing funtion on $(1, +\infty)$. In particular,

$$f(x) = \frac{x^p}{p} - x + \frac{1}{q} > f(1) = 0,$$

whenever $x > 1$, as we wanted to show.
(b) If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p \, \mathrm{d}\alpha = 1 = \int_a^b g^q \, \mathrm{d}\alpha,$$

then it follows from the previous item that

$$\int_a^b fg \, \mathrm{d}\alpha \leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q} \right) \, \mathrm{d}\alpha = \frac{1}{p} + \frac{1}{q} = 1.$$

(c) If f and g are complex functions in $\mathcal{R}(\alpha)$. Assume without loss of generality that

$$\int_a^b |f|^p \, \mathrm{d}\alpha > 0, \quad \int_a^b |g|^q \, \mathrm{d}\alpha > 0.$$

Then let

$$\tilde{f} = \frac{|f|}{\left(\int_a^b |f|^p \, \mathrm{d}\alpha\right)^{1/p}}, \quad \tilde{g} = \frac{|g|}{\left(\int_a^b |g|^q \, \mathrm{d}\alpha\right)^{1/q}}.$$

We have $\tilde{f} \in \mathcal{R}(\alpha)$, $\tilde{g} \in \mathcal{R}(\alpha)$, $\tilde{f} \geq 0$, $\tilde{g} \geq 0$, and

$$\int_a^b \tilde{f}^p \, \mathrm{d}\alpha = 1 = \int_a^b \tilde{g}^q \, \mathrm{d}\alpha,$$

so it follows from the previous item that

$$\int_a^b \tilde{f}\tilde{g} \, \mathrm{d}\alpha \leq 1,$$

which implies

$$\left| \int_a^b fg \, \mathrm{d}\alpha \right| \leq \int_a^b |f||g| \, \mathrm{d}\alpha \leq \left\{ \int_a^b |f|^p \, \mathrm{d}\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q \, \mathrm{d}\alpha \right\}^{1/q}.$$

(d) By the definitions given in problems 7 and 8, the result follows trivially.

Chapter 6, problem 11. Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 d\alpha \right\}^{1/2}.$$

Suppose that $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwartz inequality, as in the proof of Theorem 1.37.

Solution.
We have

$$\|f-h\|_2^2 = \int_a^b |f-h|^2 d\alpha \le \int_a^b (|f-g|+|g-h|)^2 d\alpha$$

$$\Rightarrow \|f-h\|_2^2 \le \int_a^b (|f-g|^2+2|f-g||g-h|+|g-h|^2)d\alpha = \|f-g\|_2^2+2\int_a^b |f-g||g-h|d\alpha+\|g-h\|_2^2,$$

but it follows from the Schwartz inequality that

$$\int_a^b |f-g||g-h|d\alpha \le \left\{\int_a^b |f-g|^2 d\alpha\right\}^{1/2} + \left\{\int_a^b |g-h|^2 d\alpha\right\}^{1/2}.$$

So

$$\|f-h\|_2^2 \le \|f-g\|_2^2 + 2\|f-g\|_2\|g-h\|_2 + \|g-h\|_2^2 = (\|f-g\|_2 + \|g-h\|_2)^2.$$

Chapter 7, problem 2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n+g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $f_n g_n\}$ converges uniformly on E.

Solution.
Assume $f_n \to f$ uniformly and $g_n \to g$ uniformly. Then given $\epsilon > 0$, there exists N_1 and N_2 such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \ge N_1, x \in E,$$
$$|g_n(x) - g(x)| < \epsilon \quad \forall n \ge N_2, x \in E.$$

So

$$|f_n(x)+g_n(x)-f(x)-g(x)| \le |f_n(x)-f(x)|+|g_n(x)-g(x)| < \epsilon+\epsilon = 2\epsilon, \quad \forall n \ge N, x \in E,$$

where $N = \max\{N_1, N_2\}$. Hence $(f_n+g_n) \to (f+g)$ uniformly.
Now assume that there exists M_n and K_n such that

$$|f_n(x)| \le M_n \quad \forall x \in E,$$
$$|g_n(x)| \le K_n \quad \forall x \in E.$$

Then first we see that, since $f_n \to f$ uniformly and $g_n \to g$ uniformly, there exists N_1 and N_2 such that

$$|f_n(x) - f_m(x)| < 1 \quad \forall n, m \ge N_1, x \in E,$$
$$|g_n(x) - g_m(x)| < 1 \quad \forall n, m \ge N_2, x \in E.$$

Let $N = \max\{N_1, N_2\}$. We have

$$|f_n(x)| \le |f_N(x)| + |f_n(x) - f_N(x)| \le M_N + 1, n \ge N,$$

so

$$|f_n(x)| \le M = \max\{M_1, M_2, ..., M_{N-1}, M_N + 1\}, \quad \forall n, x \in E.$$

Similarly, one can show

$$|g_n(x)| \le K = \max\{K_1, K_2, ..., K_{N-1}, K_N + 1\}, \quad \forall n, x \in E.$$

In particular, $|f(x)| \leq M$ and $|g(x)| \leq K$ for all $x \in E$.
Now we have

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)(g_n(x) - g(x)) + g(x)(f_n(x) - f(x))| \\ \Rightarrow |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)| \\ \Rightarrow |f_n(x)g_n(x) - f(x)g(x)| &\leq M|g_n(x) - g(x)| + K|f_n(x) - f(x)|, \end{aligned}$$

but since $M, K < \infty$ and $|g_n - g| \to 0$ and $|f_n - f| \to 0$ uniformly, it follows from the inequality above that $|f_n g_n - fg| \to 0$ uniformly, ie, $f_n g_n \to fg$ uniformly.

Chapter 7, problem 5. Let

$$f_n(x) = \begin{cases} 0 & \left(x < \frac{1}{n+1}\right), \\ \sin^2 \frac{\pi}{x} & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n}\right), \\ 0 & \left(\frac{1}{n} < x\right). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

Solution.
Clearly we see that $f_n(x) \to 0$ for all x, since for any $x > 0$ there exists N such that

$$\frac{1}{n} \leq \frac{1}{N} < x, \quad n \geq N.$$

If $x \leq 0$, then $f_n(x) = 0$ for all n.
Now given $\epsilon > 0$, let $x_n = \dfrac{2}{2n+1}$. Then we see that

$$\frac{1}{n+1} \leq \frac{2}{2n+1} \leq \frac{1}{n},$$

so

$$f_n(x_n) = \sin^2\left(\frac{(2n+1)\pi}{2}\right) = 1, \quad \forall n.$$

Therefore, since n is arbitrary, $\{f_n\}$ does not converge uniformly to 0.
We see that if $x \geq 1$ or $x \leq 0$, then $f_n(x) = 0$ for all n, and if $0 < x < 1$, there exists at most two $n's$ such that $\dfrac{1}{n+1} \leq x \leq \dfrac{1}{n}$, in this case $x = \frac{1}{k}$ for some $k \in \mathbb{N}$.
Hence trivially we see that $\sum f_n$ is convergent, in particular absolute convergent since $f_n \geq 0$. But as we saw previously f_n does not converge uniformly.