HOMEWORK #13 - MA 504

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Chapter 7, problem 1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Solution.
We showed this in the solution of problem #2, chapter 7, homework #12.

Chapter 7, problem 7. For $n = 1, 2, 3, ...,$ x real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f, and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

Solution.
We see that the parabola $nx^2 - 2\sqrt{n}x + 1$ has concave up and only one root, so $nx^2 - 2\sqrt{n}x + 1 \geq 0$ for all x real. Hence

$$f_n(x) = \frac{x}{1 + nx^2} \leq \frac{1}{2\sqrt{n}},$$

se it is trivial to see that f_n converges uniformly to 0. We have that

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2} \leq \frac{1}{1 + nx^2},$$

so if $x \neq 0$, then

$$\lim_{n\to\infty} f_n'(x) = 0,$$

but $f_n'(0) = 1$ for all n.

Chapter 7, problem 9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

Solution.
Assume that $x_n \to x$. Since $f_n \to f$ uniformly, given $\epsilon > 0$, there exits N such that

$$|f_n(y) - f(y)| < \epsilon, \quad \forall n \geq N, y \in E.$$

By the continuity of f_N there exits $\delta > 0$ such that

$$|f_N(x) - f_N(y)| < \epsilon, \quad |x - y| < \delta.$$

So

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < 3\epsilon, \quad |x - y| < \delta.$$

In particular, since $\epsilon > 0$ is arbitrary, this implies that f is continuous. Hence

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \epsilon + 3\epsilon = 4\epsilon, \quad n \geq N, |x_n - x| < \delta.$$

Therefore

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$.
The converse is NOT true. Consider $f_n(x) = \dfrac{1}{nx+1}$. So if $x_n \to x$, $f_n(x_n) \to 0$ if $x \neq 0$ and $f_n(x_n) = 1$ if $x = 0$. But we see that the convergence is not uniform, since f_n is continuous and converges to a function that is not continuous.

Chapter 7, problem 16. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

Solution.
Assume that $f_n \to f$ pointwise on K. Given $\epsilon > 0$, since $\{f_n\}$ is an equicontinuous sequence of functions, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \epsilon \quad |x - y| < \delta, \forall n.$$

We see that the balls $B_\delta(x) = \{y \in K : |x - y| < \delta\}$ cover K, and since K is compact, there exists finitely many balls, say $B_1 = B_\delta(x_1), B_2 = B_\delta(x_2), ..., B_l(x_l)$ that cover K.
So given $x \in K$, there exists x_j such that $x \in B_j = B_\delta(x_j)$, hence

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_j)| + |f_n(x_j) - f_m(x_j)| + |f_m(x_j) + f_m(x)|.$$

Since f_n converges pointwise, $f_n(x_j)$ is a Cauchy sequence. So there exits N_j such that

$$|f_n(x_j) - f_m(x_j)| < \epsilon, \quad n, m \geq N_j.$$

Let $N = \max\{N_1, ..., N_l\}$, we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_j)| + |f_n(x_j) - f_m(x_j)| + |f_m(x_j) + f_m(x)| < 3\epsilon, \quad |x - x_j| < \delta, n, m \geq N,$$

ie,

$$|f_n(x) - f_m(x)| < 3\epsilon, \quad \forall n, m \geq N, \forall x \in K.$$

Therefore $\{f_n\}$ is uniform Cauchy so it is uniformly convergent on K.

Chapter 7, problem 18. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)dt \quad (a \le x \le b).$$

Prove that there existsa subsequence $\{F_{nk}\}$ which converges uniformly on $[a, b]$.

Solution.
Assume $|f_n(x)| < M$ for all $x \in [a, b]$ and for all n. We have

$$|F_n(x) - F_n(y)| = \left| \int_y^x f_n(t)dt \right| \le M|x - y| \le M(b - a).$$

So we see that $\{F_n\}$ is a uniformly bounded equicontinuous sequence of functions. By Theorem 7.23 there exists a subsequence $\{F_{nk}\}$ that is pointwise convergent. Since $\{F_{nk}\}$ is equicontinuous, it follows from the previous problem (Chapter 7, problem 16) that $\{F_{nk}\}$ converges uniformly on $[a, b]$.

Chapter 7, problem 20. If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)x^n dx = 0 \quad (n = 0, 1, 2, ...),$$

prove that $f(x) = 0$ on $[0, 1]$. Hint: The integral of the product of f with any polynomial is zero. Use Weierstrass theorem to show that $\int_0^1 f^2(x)dx = 0$.

Solution.
First we see that if $p(x) = a_0 + a_1 x + ... + a_n x^n$ is a polynomial, then

$$\int_0^1 f(x)p(x)dx = \sum_{j=0}^{n} a_j \int_0^1 f(x)x^j dx = 0.$$

Now by Weierstraβ theorem (7.26), there exists a sequence of polynomial p_n such that $p_n \to f$ uniformly. So it follows from theorem 7.16 that

$$\int_0^1 f^2(x)dx = \int_0^1 f(x)(\lim_{n\to\infty} p_n(x))dx = \lim_{n\to\infty} \int_0^1 f(x)p_n(x)dx = 0.$$