

A singular limit in combustion: fine properties of the free boundary

G. S. Weiss*

Graduate School of Mathematical Sciences,
University of Tokyo, 3-8-1 Komaba,
Meguro-ku, Tokyo-to, 153 Japan

In this talk I present results concerning the problem

$$u \geq 0, \partial_t u - \Delta u = 0 \text{ in } \{u > 0\}, |\nabla u| = 1 \text{ on } \partial\{u > 0\} \quad (1)$$

which has been used as a simple model for flame propagation ([3]). Here $u = \lambda(T_c - T)$, T is the temperature outside the flame, T_c is the flame temperature which is assumed to be constant and λ is a normalization factor.

When dealing with equation (1) one faces the following problems: in general the solution is not unique, after finite time singularities arise and the multiplicity of the interface (i.e. the free boundary $\partial\{u > 0\}$) may increase. This is reminiscent of the problems known in the case of motion by mean curvature and the equations are indeed intrinsically related. The existence of degenerate points, i.e. points at which $\nabla u = 0$, complicates the matter further.

Let me shortly summarize the known results for equation (1): For the stationary problem Alt and Caffarelli obtained existence of a solution by minimizing the energy $E(v) := \int (|\nabla v|^2 + \chi_{\{v>0\}})$ on an affine subspace of $H^{1,2}(\Omega)$ ([1]). They proved furthermore that the minimizer is locally Lipschitz continuous, that it is a solution in the sense of distributions and that the free boundary is regular up to a set of vanishing \mathcal{H}^{n-1} -measure. In [5] I showed that the singularities are isolated for $n = 3$ and that the singularities are a $n - 3$ -dimensional set in general.

For the application in combustion theory a natural approximation to equation (1) turns out to be

$$\begin{aligned} \partial_t u_\epsilon - \Delta u_\epsilon &= -\beta_\epsilon(u_\epsilon), \quad \beta_\epsilon(s) = \frac{1}{\epsilon} \beta\left(\frac{s}{\epsilon}\right), \\ \beta &\in C_0^\infty([0, 1]), \quad \beta > 0 \text{ in } (0, 1), \quad \int \beta = \frac{1}{2}. \end{aligned}$$

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Concerning this approximation, Berestycki, Caffarelli and Nirenberg ([2]) proved in the stationary case that u_ϵ is uniformly Lipschitz continuous. Assuming the existence of a *minimal solution* of the stationary equation they could prove further steps towards convergence to the limit problem.

In the case of time dependence, Caffarelli and Vazquez ([4]) showed that u_ϵ is uniformly Lipschitz continuous. For “strictly concave” initial data they proved that any limit is a solution in the sense of distributions.

For the two-phase case there are results by Caffarelli-Lederman-Wolanski, D. Danielli and others.

The results I present in my talk on the problem with general initial data in higher dimensions include the following: each limit of u_ϵ is a solution in the sense of domain variations. For a.e. time the non-degenerate singular set is $n - 1$ countably rectifiable.

References

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