Basic set notes

Given two sets X, Y, we can form a function

$$f\colon X \to Y$$

which is an assignment to each x in X, a unique y in Y denoted $f(x) = y$. We say the function is **injective** or 1-1 if when $f(x_1) = f(x_2)$, then $x_1 = x_2$. We say it is **surjective** or onto if for each $y \in Y$, there exists $x \in X$ such that $f(x) = y$. We call X is the **domain** of the function, Y the **codomain**, and

$$f(X) = \{f(x) \in Y \ : \ x \in X\}$$

the **range** of X. For a subset $S \subset X$, the **image** of S is

$$f(S) = \{f(s) \ : \ s \in S\} \subset Y.$$

For a subset $T \subset Y$, we define the **pullback** of T to be

$$f^{-1}(T) = \{x \in X \ : \ f(x) \in T\} \subset X.$$

The **union** of two sets $X, Y \subset \Omega$ is denoted $X \cup Y$ and is

$$X \cup Y = \{z \in \Omega \ : \ z \in X \text{ or } z \in Y\}$$

and the **intersection** $X \cap Y$ is

$$X \cap Y = \{z \in \Omega \ : \ z \in X \text{ and } z \in Y\}.$$

We say X, Y are **disjoint** if $X \cap Y = \emptyset$. The **product** of two sets X, Y is the set

$$X \times Y = \{(x, y) \ : \ x \in X, \ y \in Y\}.$$

If X is a finite set, $|X|$ denotes the cardinality or number of elements in X. We say two sets X, Y have the same cardinality if there is a bijective function

$$f\colon X \longrightarrow Y.$$

For finite sets, the following statement is sometimes called the **Pigeonhole Principle**:

Let $f\colon X \to X$ be a function and $|X|$ is finite. Then f is injective if and only if f is surjective.

Proof. To begin, we first notice that

$$|X| = \sum_{y \in f(X)} \left| f^{-1}(y) \right|. \tag{1}$$

To see this, notice that

$$X = \bigcup_{y \in f(X)} f^{-1}(y).$$

As f is a function, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint if $y_1 \neq y_2$. Altogether, we get (1).

Now assume that f is injective. Then $\left|f^{-1}(y)\right| = 1$ for every $y \in f(X)$. Thus by (1)

$$|X| = \sum_{y \in f(X)} \left|f^{-1}(y)\right| = \sum_{y \in f(X)} 1 = |f(X)|.$$

Since $f(X)$ is a subset of X and they have the same size, they must be equal. In particular, f is surjective.

Now assume that f is surjective. Then $\left|f^{-1}(y)\right| \geq 1$ for every $y \in X$. Thus by (1), we have

$$|X| = \sum_{y \in X} \left|f^{-1}(y)\right| \geq \sum_{y \in X} 1 = |X|.$$

In particular, if $\left|f^{-1}(y)\right| > 1$ for some y, then we have

$$|X| > |X|,$$

which is impossible. So $\left|f^{-1}(y)\right| = 1$ for all $y \in X$ and thus f is injective. □

A standard corollary of the Pigeonhole Principle is the following basic statement: If I have n marbles and m bags, if I place each marble in a bag and $n > m$, then there must be a bag with at least two marbles.

The mathematical statement is: If $f \colon X \to Y$ is a function and $|X| > |Y|$, then f cannot be injective. (Note, the map need not be surjective as I could simply put ALL the marbles in one bag).

When we have functions $f, g \colon X \to \mathbf{R}$, we can define a pair of new functions:

$$f + g, f \cdot g \colon X \longrightarrow \mathbf{R}$$

by

$$(f+g)(x) = f(x) + g(x), \quad (f \cdot g)(x) = f(x)g(x).$$

In fact, we only need a field F for the codomain. Thus

$$f + g, f \cdot g \colon X \longrightarrow F$$

make sense when F is a field. When the codomain is an F–vector space V, we can define

$$f + g, \lambda f \colon X \longrightarrow V, \quad \lambda \in F$$

by

$$(f+g)(x) = f(x) + g(x), \quad (\lambda f)(x) = \lambda f(x).$$

When we have functions $f \colon X \to Y$ and $g \colon Y \to Z$, then we can define the **composition** $g \circ f \colon X \to Z$ by

$$(g \circ f)(x) = g(f(x)).$$