Name: Solutions Math 511 March 10, 2010

(Work will be graded on the basis of clarity as well as accuracy.)

(15) 1. A mathematical statement is "True" if it is true in all cases possible; otherwise it is 'False.' Indicate T or F for the following assertions *and indicate a reson to justify your answer*:

(a) If A is a 4×3 matrix, then the equation $Ax = 0$ has at least one solution.

T, A homogeneous equation always has the solution $x = 0$

(b) If A is an $m \times n$ matrix with linearly independent columns, then it has linearly independent rows.

False - true if $m = n$. Ex $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(c) The solutions to the equation $Ax = 0$ form a vector space.

True: $Ax_1 = 0$, $Ax_2 = 0 \Rightarrow A(cx_1 + dx_2) = 0$

(d) If all diagonal entries of A are zero, then A is singular.

F : $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(e) If P is the elementary matrix which exchanges rows i and j $(i \neq j)$, then det P=1.

F : $\det P = -1$

(10) 2(a). A set $\{v_1, v_2, \ldots, v_k\}$ is **linearly independent** if: (use the space to complete the sentence)

whenever $\sum \lambda_i v_i = 0$ we must have $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$

(b) Let v_1, v_2 and v_3 be orthogonal. Prove that they are linearly independent.

Let $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$.

Show $c_2 = 0$, for example. Well

$(c_1 v_1 + c_2 v_2 + c_3 v_3, v_2) = (0, v_2) = 0$

$= c_1 \underbrace{(v_1, v_2)}_{=0} + c_2 (v_2, v_2) + c_3 \underbrace{(v_3, v_2)}_{=0} = 0$ since $(v_1, v_2) = (v_3, v_2) = 0$

Thus $0 = c_2 (v_2, v_2)$. But $v_2 \neq 0$

$= c_2 |v_2|^2$.

So $c_2 = 0$.

(20) 3. Let T be the matrix on $\mathbb{R}^2$ with $Te_1 = (2,4)$ and $Te_2 = (-1,1)$ (e_1 and e_2 are the usual basis vectors in the plane).

(a) Write down the matrix A which gives T with respect to this basis.

$A : \begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix}$

$A =$

(b) Find A^{-1}, and use it to determine the preimage of $(1,1)$ under T.

A^{-1}: by formula:

$A^{-1} = \frac{1}{6}\begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/6 & 2/3 \\ -1/6 & 1/3 \end{pmatrix}$.

To find premage of $(1,1)$, we take $A^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1/6 & 2/3 \\ -1/6 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix}$

Let's check: we need $A\begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix}$

(20) 4. (a) Give a basis for the column space C and null space N of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}.$$

Basis of C is $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$; better $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ There are many choices, but this seems to have least redundancy.
The null-space is 1-dimensional with basis $(1,-1,0)$

(b) Show that ~~C~~ $C(A)$ and ~~N~~ $N(A^T)$ are orthogonal, and explain why in general they are *orthogonal complements* of each other (the word *rank* should appear somewhere in your discussion). My error.

Here is where there was some misunderstanding.
To find $N(A^T)$, we look at

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

and find that (starting from bottom) a basis is

$(0, 0, -\frac{1}{2}, 1)$ and $(-\frac{1}{2}, 1, 0, 0)$

(the free variables are in columns 2 and 4)

Notice the word <u>why</u> which is circled. That means one lost points if we just check that the basis vectors in $N(A^T)$ are orthogonal to those in C. You were to explain, in your own words, the main results of section 3.1,

(20) 5. Let A be an $m \times n$ matrix with linearly independent columns, and consider the equation $Ax = b$.

(a) In this situation (for general m and n) explain why we expect this equation not have a solution.

More equations than unknowns.

(b) Let $\hat{x}$ be the 'solution' given by least squares. Derive the formulas for $\hat{x}$ and $A\hat{x}$ using various matrices associated to A, etc. What is the relation between $b - A\hat{x}$ and the range of A (this can almost be answered in one word)? — they are orthogonal

$$\hat{x} = (A^TA)^{-1}A^Tb$$
$$A\hat{x} = A(A^TA)^{-1}A^Tb$$

(c) If we consider the least squares solution to the system $Cx + D = b$ with data $x = -1, 0, 1$ and the b-values $3, 2, 2$, what is the matrix $\cancel{C}$ A. Does $\cancel{C}$ A have linearly independent columns? Linearly independent rows?

Yes No

$$A\begin{pmatrix} C \\ D \end{pmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \end{pmatrix}(= \hat{x}) = \left(\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\right)^{-1}\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

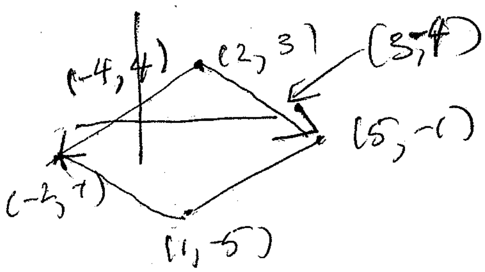
$$= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{-1}\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 7/3 \end{bmatrix}$$

(15) 6. (a) Find the area (using what we learned in class) of the parallelogram with vertices (2,3), (5, -1), (-2,-1) and (1,-5).

$$\left| \det \begin{vmatrix} 3 & 4 \\ -4 & 4 \end{vmatrix} \right| = 28$$



(b) Suppose we have a 6×6 matrix A, and we know all its cofactors. How would you find the entry a_{12} of A? (Bad problem)