Please study

Name: SOLUTIONS **Math 511 April 22, 2010**

(Work will be graded on the basis of clarity as well as accuracy.)

(25) 1. Let A be a positive definite matrix and $x_1, \ldots x_n$ an orthogonal basis with respect to which A is diagonal. Let C be nonsingular and define

$$B = C^T AC.$$

Note this does NOT say B is similar to A. That would mean $B = C^{-1}AC$

(a) Using any appropriate criterion, show that B is positive definite.

Easiest way! Show if $y \neq 0$, $y^TBy > 0$. But $y^TBy = y^TC^TACy = (Cy)^TACy$ and $Cy \neq 0$ since C is nonsingular!

People write about X^TBX, but things like that are not in our discussion of positive definite.

Note If you show that B is symmetric and $\det B > 0$, you have not shown B is p.d. Ex $\begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$.

(b) Suppose we have $x_j = Cy_j$ where C is from (a). Must the $\{y_j\}$ also be orthogonal?, Linearly independent? Justify.

Here people gave lots of hand-waving. The y's are lin ind, but no reason to be orthogonal.

Orthogonal? All we are saying is that C is nonsingular. So C could take i to i and j to $i+j$. Then C_j and C_i are NOT orthogonal!

Lin ind? Suppose $\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_n y_n = 0$

Then the matrix $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$ has rank $< n$. But $Y = XC^{-1}$ (here Y and X are right) and X and C are nonsingular, $\det XC^{-1} \neq 0$ since X and C are nonsingular.

So $\det Y$ can't be 0!

Note Several students said the y's were orthogonal but not linearly independent. Get clear at that please!

(*c*) Suppose that A is the matrix

So that A and B are not similar,

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Some people said B was pos def !! But then 0 is a eigenvalue.

What can you say about the eigen**values** of B, with B defined in (*a*)?

All we can say (p 324) is that B has one 0 eigenvalue, one >0 eigenvalue. The book (and lecturer) should have made this clearer. I made this example $C = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$. I then get $B = \begin{bmatrix} 84 & 62 \\ 62 & 46 \end{bmatrix}$, so char eqn is $\lambda^2 - 130\lambda + 20$. So 1, 5 are NOT eigenvalues.

(25) 2. (*a*) Show that the matrix

$$A = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

is positive definite (use any criterion that works for you).

you should mention that A is symmetric

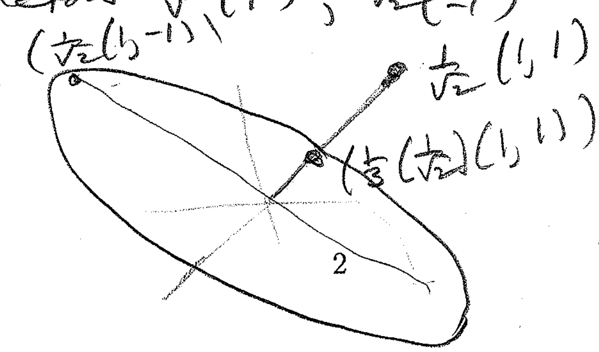
Then principal minors are 5, 9

(*b*) Sketch the ellipse $5x^2 + 8xy + 5y^2 = 1$ in the xy–plane.

$\lambda = 9$ eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda = 1$, e-vector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(unit evectors $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$)

ellipse



(*c*) Write $A = U\Lambda U^T$ with U unitary and Λ diagonal (so show the matrices U and Λ).

evector for 1 evector for 9

$$U: \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$$

(*d*) Can we choose U to be triangular in this sitation? Explain.

(Bad question. Answer was meant to be "no" since the eigenvectors won't be triangular)

(*e*) Find the max and min of the Rayleigh quotients $x^T Ax/x^T x$.

9 is max (largest eigenvalue)
1 is smallest (least eigenvalue)

(20) 3. Let A the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(*a*) Find the eigenvalues of A: A is Δlar so evalues are $0, 0, 1$

(*b*) From the information in (*a*) write down the possible Jordan form J that A might have (arrange this so that the diagonal elements of J are nonincreasing).

0 block can be $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. 1-block has to be $[1]$

possibilities are then $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (a diagonal matrix is certainly a good Jordan one)

(*c*) Which of the possibilities in (*b*) is the correct Jordan form, and explain ~~why~~.

We have to work a bit. The eigenspace for $\lambda = 1$ has dimension 1, but for $\lambda = 0$ it is not clear. But the matrix A has rank 1, so its nullspace has dimension 2 — in fact, basis vectors for 0-eigenspace are $\underset{x_1}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$ and $\underset{x_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$. For 1-space it is $\underset{x_3}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

(*d*) Let x_1, x_2, x_3 be the basis for which J has the form in (*c*). Is this guaranteed to be orthogonal? Explain (you are not expected to produce this basis!).

In this example $x_1 \not\perp x_2$, $x_2 \perp x_3$ but that is all. But we should think about it.

For the 0-space, we know the nullspace has dimension 2. But there IS no reason for any two vectors in the nullspace to be $\perp$ (here they are, though, and by Gram-Schmidt we could arrange that). But the vectors in the 1-subspace are only linearly independent from these in the 0-subspace (check the book!). So no reason to expect orthogonal.