MA 511 ASSIGNMENT SHEET Spring(!) 2010

Text: G. Strang *Linear Algebra and its Applications*, Fourth Edition

Warning: the problem numbers are keyed to the fourth edition (hard-cover, 2006). I believe there is also an international edition with the same material, but the problem numbers are **not** the same. Be sure that you use the numberings of this edition, and a copy is on reserve in the Mathematics Library.

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But ! take advantage of email for questions about lecture, homework, etc. since this is a large class, we will not be able to have a very good percent of the homework graded, and one thing we can do is have volunteers write their solutions to more difficult problems; I can do this too on occasion, but you should let me know which problems need help. I would expect our web page to be a good place to check (from my section).

Even if you have some familiarity with this material, be worned that we will be trying to keep a very contemporary point of view, and worry and efficiency of computing, error analysis, etc. Unfortunately, your instructor is a pure mathematician, and so he would appreciate observations from class members; we should probably clear them in advance via email.

April 13, 2010 version

We hope to cover most of the text, and in particular give full attention to some interesting applications. This sheet will be updated throughout the semester.

The course will move fast, and it is important to come to **every** *class.* The book is written in a very informal way, and unless you read it very critically you will have difficulty understanding what the author is saying; my job is partly to help you in this. The author has good summaries in the text, but you might slide over them — read carefully. And unlike in many magthematics classes, we will pay attention to the number of computations and how *robust* our methods are: this is an important issue in applications even if it is not always noticed in courses. If we change the information 'a little' will the answer change very much?

Some of the homework problems have answers/solutions in the back. There are far too many problems for us to penetrate a good percentage, but there are lots of opportunities for you to work out problems on your own.

I plan two major exams (evening) and several quizzes, usually unannounced. The purpose of the quizzes is to help the class keep to date, but the class size may the frequency.

We will learn soon that matrix multiplication is not commutative. So we usually think of vectors as column vectors, and so write $A\mathbf{x}$ for the action of A on the vector $\mathbf{x}$. That means that rows and columns will play different roles. (And we will often follow the author's laziness and not slavishly use different fonts for vectors and scalars.)

1.1-1.3 *Systems of Linear Equations.* (**1/12**) This introduces the basic framework and reinforces the value of a geometric viewpoint, as well as simply computing. This will also show ho we must be careful in formulating statements. How do we 'count' equations? Is the system $2x = 3, 4x = 6$ one or two equations? How do we tell? How about $2x = 3, 4x = 7$?

From high school we 'know' that n 'linear' equations in n unknowns 'has' one and only one solution, but in fact this is not always true — it depends on how you count! We view such a system as either n linear equations with real numbers unknown or as a single equation in n-dimensional vectors; obviously the second way will be easier as we encounter more cumbersome situations (but many things depend on huge numbers of variables!). We introduce Gaussian elimination and address the *efficiency* of this algorithm. Later we will learn about the *rank* of a matrix and augmented matrix, and this is how we 'count' the number of equations in a system.
Problems: p. 9: 2, 3, 4, 5; p. 15: 6, 11, 18.

1.4 *Matrices and their algebra.* (**1/14**) Matrices are an efficient may to express systems of equations in a way to which humans can relate. *Elementary matrices* give an algebraic way to view Gaussian elimination. **Problems:** p. 26: 2, 3(a), 7, 20, 24

1.5 *Triangular factorization.* (**1/19**) Decomposition $A = LU$ or (more symmetrically) $A = LDU$ (p. 36) *if no row exchanges are necessary.* Otherwise, need to apply principle to PA instead of A, where P is a permutation matrix, and $P^{-1} = P^T$. **Problems:** p. 39: 1, 5, 6, 8, 13, 21

1.6 *Inverses, symmetric matrices.* (**1/19**) **Problems:** p. 52: 5, 6, 11 (a, b), 13 , 25, 30

1.7 *Certain applications, large simple matrices, conditioning.* (**1/21**) **Problems:** *added 1/20:* No homeowrk for section 1.7.

Review: p. 65: 12, 19, 22, 26a,b

2.1 *Vector spaces (subspaces).* (**1/26**) Closed under + and scalar multiplication. This is where you should be clear on the definition: some strange objects can be vector spaces. Contrast *subspace* and *subset*. Two important subspaces arise in solving systems of linear equations: the nullspace and the column space; be sure that you can make clear sentences about solving linear equations in terms of these (sub)spaces. **Problems:** p. 73: 2, 3, 7 (a, b, c), 11, 15

2.2 $A\mathbf{x} = \mathbf{b}$ *in the general case.*({bf 1/27} although initiated 1/26) Echelon, row(-reduced) echelon form, pivot and free variables. Note procedures outlined informally on pp. 80 and 83. **Problems:** p. 85: 2, 4, 6, 11.

2.3 *Linear independence, spanning set, basis, dimension.* (**1/28**) Another way to count vectors. **Problems:** p. 98: 1, 6, 14, 18, 19, 20, 24, 35, 38

2.4 *Fundamental (sub)spaces of an $(m \times n)$ matrix A.* (**2/2**) • **This is a little complicated to keep in mind!**. This important section connects many of the ideas we have introduced. Column space (dim r), nullspace (dim $n-r$), row space (sol space of A^T), left nullspace (nullspace of A^T). (The first two are in $\mathbb{R}^m$; the other two in $\mathbb{R}^n$.) These are related to the echelon forms U and (reduced echelon) R of A. The *fundamental theorem of linear algebra* is on p. 106. We learn about left/right inverses. **Problems:** p. 110: 3, 4, 7, 11, 21, 27; p. 137: 2, 5, 11

2.6 *Linear Transformations.* (**2/4**) Definition: $T(c\mathbf{x}+d\mathbf{y}) = cT(\mathbf{x}) + dT(\mathbf{y})$. Examples come from matrix algebra and also from 'function spaces,' operations such as $f \to f'$, $f \to \int_0^x f(t)\,dt$. Determined by action on a basis (however, $(x+y)^2 \neq x^2+y^2$!). Special matrices: P (projection), Q (rotation), H (reflection). On pages 128,129 we find how to associate a matrix to a linear transformation T. Is there only one matric for each transformation? How about for the important example $Tp = p'$ for $p \in \mathbf{P}_4$? **Problems:** p. 133: 4, 6, 7; p. 137: 29, 31.

3.1 **Lengths, Angles, Orthogonality** (**2/9**) Orthogonal complements. Fundamental theorem of orthogonality (p. 144). Reinterpret fundamental theorem of linear algebra. **Problems:** p. 148: 2, 3, 11, 14, 19.

3.2 *Cosine!* (**2/11**)is more important than sin. Projection onto a subspace (high-school math helps here). Projection: $P^2 = P$. Note: sometimes I write (x, y) instead of xy^T (which the book uses). **Problems:** p. 157: 3, 5, 10, 12, 17.

3.3 *Least squares.* (**2/16**, **2/18**) Find 'best' solution to $A\mathbf{x} = \mathbf{b}$ with b confined to a subspace S. If $e(\subset S)$ is this solution so that $e = A\hat{x}$, then e is perpendicular to S. The number of applications of this section is a course in itself, we just skim the surface. Notice how the formula $P = A(A^TA)^{-1}A^T$ is the only way we can write the formula from the previous section–there A was a column vector a, now it is an $m \times n$ matrix, with $m > n$. **Warning:** The author usually requires that the columns of A be independent, that is when the matrix A^TA has an inverse.

The important formulas are on p. 162–note the distinction between the *best estimate* $\hat{x} = (A^TA)^{-1}b$ and the projection, which is $A\hat{x}$. **Problems:** p. 170: 3, 4 (think about why calculus is relevant here!), 22, 23, 31; **also** 7, 14, 17, 18, 27 (these are due 2/23).

3.4 *Orthogonal matrices and bases.* (**start 2/18**) (Book notes that 'orthonormal' matrix would be a better term.) If Q is orthogonal *and square!* then $Q^{-1} = Q^T$. Q **is reserved for matrices with orthonormal columns** . If Q is square, every vector may be written as a linear combination of the columns (or rows!) of Q, and we get a formula for the coefficients: if $x = \sum x_j q_j$, then $x_j = q_j^T x$, where q_j is the jth row of Q.

If Q is not square (so it will '! have to ! have more rows than columns), then we want the best (least-squares) solution to $Qx = b$. The Gram–Schmidt process transforms any linearly independent set of vectors into an equivalent (what do we mean by that?) set of *othonormal* vectors. **Problems:** p. 185: 1, 3, 6, 11, 14.

3.4′ (**2/23**) Now factor a matrix A as $A = QR$; Q is orthogonal and R is right-triangular (not quite the U we had in Chapter 1), and R is invertible. See the formula (12) on p. 181:

$$A = \begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} q_1^T & q_1^T b & q_1^T c \\ & q_2^T b & q_2^c \\ & & q_3^T c \end{pmatrix} = QR.$$

We apply these ideas to vector spaces of *functions* and see that expanding a function in a Fourier series is just the Gramm-Schmidt process. (Don't be terrified by this, it just uses the formulas we've been developing.) Best linear fit for data.

Problems: p. 187, 16, 21, 25, 29, 31.

End of Chapter 3 for us **unless there is strong desire to use FFT** .

4.1 -4.3 Determinants (**3/25** for24.1; the other on **3/2**). We follow the text and define the deerminant of an $n \times n$ matrix A, *det*(A) or $|A|$, as a function of A which is 1 for the identity matrix, changes sign when two rows are interchanged (this is a special kind of permutation called a transposition), and is linear with respect to operations on the first row. Of course, this has many consequences, which are points 3–10 in §4.1 of the book. Be a little careful (!), since sometimes people learn a formula for the 3×3 determinant which doesn't work when $n > 3$. We derive several formulas for $|A|$, including the one with cofactors. In principle, $\det(A)$ involves n^n sums, but we see quickly that there are really only $n!$ sums (which is far less than n^n).

Problems :*Note: only §§4.1, 4.2 dues 3/2* p. 206: 5, 8, 15, 17(c), 28, 29; p. 215: 5, 6, 9 (a, b), 12 (challenging).

4.4 Applications of determinants (**3/4**). Formula for A^{-1} (not computationally efficient), Cramer's rule. Determinants and volumes. We answer the question directly now: when does A factor: $A = LU$?

Problems: 3, 5, 10, 14, 15, 28.

End of Chapter 4

Review: Exam March 10 (evening)

5.1 *Introduction to eigenvalues.* (**3/11**) Instead of $Ax = 0$, we now ask: when does the equation $Ax = \lambda x$ (where λ is a scalar) have a *nontrivial* solution (so that $x \neq 0$). This comes up in systems of equations, and we quickly find that for most λ there is only the trivial solution. The x for which this euqation has a solution are called *eignevectors*, and our goal is to find, as best as possible, a basis consisting only of eigenvectors (why is this a good idea?). The trace and determinant of A

are expressed in terms of the eigenvalues of A. Does a nontrivial rotation have any eigenvectors?

Problems: p. 240: 3, 4, 7, , 11, 14, 25, 26 [which is why we will be introducing complesx number fairly soon], 30.

5.2 Best case: diagonalization. If there are n linearly independent eigenvectors, then A can be diagonalized with respect to a basis consisting of eigenvectors.

The matrix S with eigenvactors as columns is said to effect a *similarity*. Eigenvectors corresponding to different eigenvalues are linearly independent.

Problems: p. 250: 4, 5, 12, 17, 24 29, 30.

5.4 e^A and stability. If we can diagonalize $A = S\Lambda S^{-1}$, then $du/dt = Au$ has the solution $u(t) = e^{At}u(0)$, and this is very easy to compute. The solutions are *stable* if $\Re\lambda_i < 0$ for all i, and unstable when at least one has positive real part.

We show how a linear equation of higher order may be written as a first-order linear system.

Problems: p. 275: 1, 3, 7, 14 [use material at bottom of p. 273 as model], 21, 22, 36.

5.5 Complex matrices. (**due 4/6**) Material to p. 282 bottom is routine. New vocabulary: Hermetian, unitary matrix (analogues of symmetric and othrogonal matrices). Good dictionary on p. 288. The famous *spectral theorem* is introduced in an offhand manner on p. 285.

Problems: 7, 8(!), 11, 14, 20.

Appendix B (Jordan form). Here we sketch what happens when the matrix A cannot be diagonalized; the Jordan form is the best we can do.

Problems: p. 427: 1(b), 6, 7.

5.6 Similarity as a subject in itself. (**Probs fhrough 15 due 4/6**)We think of a similarity matrix as expressing a change of basis (usually made so that a linear transformation is simpler to understand with respect to a different basis). We can always transform a matirx A by a unitary similarity $U^{-1}AU$ to be triangular (what happens if A is Hermetian??). Spectral theorem formally stated on p. 297. Examples for Jordan form. p. 298: Normal matrices defined: these are exactly the matrices with a full set of orthonormal eignevectors.

Problems: p. 302: 3, 5, 8, 9, 12, 15, 18, 20, 24, 35 [also find e^J], 38, 41.

3.5 *FFT* Very efficient, one of the most quoted papers on the last century.

Problems: p. 196: 1, 3, 7, 11, 15, 18, 21 [don't try this if not comfortable].

Positive Definite Matrics. Saddle points.

6.1 *(review of MA 261!, can be done via quadratic formula.)* Consider the *quadratic form* x^TAx – is it always positive? Can it change signs?? This is the second term of a Taylor series.

Problems. p. 316: 3, 4, 7, 9, 15, 20.

6.2 *Tests for P. D.* (**4/13)**
Problems. p. 326: 1, 4, 7, 10, 17, 29

6.3: *This is omitted in Spring 2010* '[a] great matrix factorization.' Any matrix A may be factored as $A = U\Sigma V^T$ where the outside factors are square. The columns of U, V come from eigenvalues of AA^T, A^TA. we touch upon some applications.

Problems. 1, 2, 6, 8, 10, 14.

6.4 *Minimum principles.* Most of this only is for positive definite matrices. We get the formula for least squares (see p. 162)

$$\hat{x} = (A^TA)^{-1}A^Tb$$

in another manner. We also consider the *Rayliegh quotient*

$$\frac{x^TAX}{x^Tx},$$

which is easy to analyse with respect to an orthogonal basis (which is always possible!).

Problems. 1, 2, 4 **these are due 4/15)**, 5, 7, 11a.

7.2 *Matrix norm, sensitivity.* **4/15)** We define the 'norm' of the matrix A as

$$\|A\| = \max |Ax|,$$

where the maximum is over vectors x of length 1. This can be written other ways. It is easy to compute for p. d. matrices; in general we replace a not-necessarily-=p. d. matrix A by A^TA or AA^T, but must remember to take square roots. The *condition number* of a p. d. matrix is

$$c = \|A\|\|A^{-1}\|$$

, and we have that $\max\lambda/\min\lambda$. This is important for computation: If in the equation $Ax = b$ we replace either b by δb or A by δA, then the errors (respectively) $x + \delta x$ or δx of the solutions are controlled by this number c (p. 355).

Problems. p. 357: 1, 5, 6, 11a, 12 bc, 13 a.