Look up details in text, too.

Name: Solutions **Math 511** **April 8, 2010**

(*Work will be graded on the basis of clarity as well as accuracy.*)

(20) 1. True-False. Write T or F and give a reason in each case If the answer is F, try to give an example.

(a) If A is an $n \times n$ matrix with n distinct eigenvalues, then A is similar to a diagonal matrix. T Ch 5

(b) If A is $n \times n$ with n distinct eigenvalues, then it is invertible.

False ; $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ has distinct e-values, but det $= 0$

(c) If $Ax = 2x$ and $Ay = 3y$ then x and y are orthogonal.

No – see problem 4. But it is true if A, B symmetric

(d) If A and B are similar than they have the same eigenvalues.

$A = MBM^{-1}$ so $\det(A - \lambda I) = (\det M) \det(B - \lambda I) \frac{1}{\det M}$.

(e) If A and B are similar they have the same eigenvectors.

False. Eigenvectors of A are columns of M when $A = M\Lambda M^{-1}$, an eigenvectors of Λ are $e_1, \ldots e_n$.

(15) 2. Write the differential equation $y'' + 6y' - 2y = 0$ in the form $x' = Ax$ where A is a 2×2 matrix.

You have to tell me what are x and A.

$x \doteq \begin{pmatrix} y \\ y' \end{pmatrix} \quad x' = \begin{pmatrix} y' \\ y'' \end{pmatrix}$. So

$x' = \begin{pmatrix} 2 & -6 \\ 0 & 1 \end{pmatrix} x$

(15) 3. Let V be the vector space of 2×2 real symmetric matrices (so that $M^T = M$ for $M \in V$).

(*a*) What is the dimension of V; prove it by exhibiting a basis for V.

This was discussed in almost same situation in class.
Your basis has to be 2x2 matrices.
dim = 3
Basis: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
(there are only 3 independent ~~...~~ entries in any symmetric M.

(30) 4. **This question has several parts.** Let

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(*a*) Find the eigenvalues and eigenvectors of A. Show work.

evectors:
$\lambda = 0$ evector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$\lambda = -2$ evector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

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(*b*) Write all matrices S, Λ, S^{-1} which appear in the factorization $A = S\Lambda S^{-1}$ where Λ is diagonal and A is from (*a*).

$S: \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

$A = S \Lambda S^{-1}$

(*c*) What is e^{At}? How do you know it is never singular?

$e^{At} = S e^{\Lambda t} S^{-1} = S \begin{pmatrix} 1 & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

To take det e^{At}, take (simpler) product of ~~these~~ three!

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(*d*) Solve the differential equation

(1) $$x'(t) = Ax(t),$$

(same matrix A) (it will be the sum of two special solutions), and uses work from the first parts.

$$x(t) = c e^{0t}\begin{pmatrix}1\\1\end{pmatrix} + d e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}$$

(*e*) For what initial condition(s) (i. e. choice of $x(0)$) will the solution to (1) tend to zero as $t \to \infty$?

Need $c = 0$ in (d)