RANDOM WALKS IN DIRICHLET ENVIRONMENTS WITH BOUNDED JUMPS

by

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To the children God has blessed me with during my time in grad school: two lost to miscarriage, and Ezra and Rose.

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ABSTRACT

This thesis studies non-nearest-neighbor random walks in random environments (RWRE) on $\mathbb{Z}$ and $\mathbb{Z}^d$ that are drawn in an i.i.d. way according to a Dirichlet distribution. We complete a characterization of recurrence and transience in a given direction for random walks in Dirichlet environments (RWDE) by proving directional recurrence in the case where the Dirichlet parameters are balanced and the annealed drift is zero. As a step toward this, we prove a 0-1 law for directional transience of i.i.d. RWRE on $\mathbb{Z}^2$ with bounded jumps. Such a 0-1 law was proven by Zerner and Merkl for nearest-neighbor RWRE in 2001, and Zerner gave a simpler proof in 2007. We modify the latter argument to allow for bounded jumps. We then characterize ballisticity, or nonzero limiting velocity, of transient RWDE on $\mathbb{Z}$. It turns out ballisticity is controlled by two parameters, κ_0 and κ_1. The parameter κ_0, which controls finite traps, is known to characterize ballisticity for nearest-neighbor RWDE on $\mathbb{Z}^d$, $d \geq 3$, where transient walks are ballistic if and only if $\kappa_0 > 1$. The parameter κ_1, which controls large-scale backtracking, is known to characterize ballisticity for nearest-neighbor RWDE on $\mathbb{Z}$, where transient walks are ballistic if and only if $|\kappa_1| > 1$. We show that in our model, transient walks are ballistic if and only if $\min(\kappa_0, |\kappa_1|) > 1$. Our characterization is thus a mixture of known characterizations of ballisticity for nearest-neighbor one-dimensional and higher-dimensional cases. We also prove more detailed theorems that help us better understand the phenomena affecting ballisticity.

1. INTRODUCTION

This thesis studies non-nearest-neighbor random walks in random environments (RWRE) on $\mathbb{Z}$ and $\mathbb{Z}^d$. We allow jumps of bounded size, and assume transition probability vectors at various sites are drawn in an i.i.d. way. Most of our results deal random walks in Dirichlet environments (RWDE), where the transition probability vectors are drawn according to a Dirichlet distribution. Our main results concern directional recurrence and transience and limiting velocity.

1.1 Background

RWRE were first treated in depth by F. Solomon [1] in 1975, and the study has since grown in many directions. Solomon focused on nearest-neighbor RWRE on $\mathbb{Z}$, characterizing directional transience and calculating limiting speed (under mild conditions) in terms of simple, easily computable expectations involving the environment at a single site. One of the main results from Solomon's paper is that a RWRE can approach infinity almost surely, but with an almost-sure limiting speed of zero, a surprising phenomenon that cannot occur for random walks in homogenous or even periodic environments. A random walk is called *directionally transient* if it has an almost-sure limiting direction. Under appropriate i.i.d. assumptions on the environment, all directionally transient nearest-neighbor RWRE on $\mathbb{Z}^d$ have an almost-sure limiting velocity v (which can be 0). This is also true of RWRE with bounded jumps on $\mathbb{Z}$; see Appendix A. We say a directionally transient random walk is *ballistic* if the limiting velocity v is nonzero.

RWRE have proven quite challenging to analyze in settings other than the nearest-neighbor case of $\mathbb{Z}$. For instance, although some sufficient conditions for directional transience and ballisticity have been studied for nearest-neighbor RWRE on $\mathbb{Z}^d$, no general characterizations are known, and known sufficient conditions are often quite difficult or impossible to check in general (see, for example, [2]). Nevertheless, certain special cases have proven to be more tractable. Examples include random environments that almost surely have zero drift at every site (e.g., [3], [4], [5]), random environments that are small perturbations of simple random walks (e.g., [6], [7], [8], [9], [10]), and environments where transition probabilities are deterministic in some directions and random in others (e.g., [11]). The case of RWDE is another notable example. Many conjectures that remain open for general nearest-neighbor RWRE on $\mathbb{Z}^d$ have been proven for RWDE, including a characterization of directional transience for all d and a characterization of ballisticity for $d \geq 3$.

Dirichlet environments present a helpful case study for general RWRE, giving insight into what sorts of behaviors are possible. For example, a noted conjecture (see [12]) asserts that under an assumption of uniform ellipticity (where transition probabilities are bounded uniformly from below), nearest-neighbor RWRE on $\mathbb{Z}^d$, $d \geq 2$, that are directionally transient are necessarily ballistic. Certain RWDE provide a counterexample for the non–uniform elliptic case [13], showing that the uniform ellipticity assumption is necessary for this conjecture. At the same time, in the Dirichlet case, the factors that can cause ballisticity to fail are entirely due to the non–uniform ellipticity property, providing some additional evidence for the conjecture (see [14], [15, Remark 5.13]).

In some sense, the setting of RWRE on $\mathbb{Z}$ with bounded jumps—and of RWRE on a strip, a generalization of the bounded-jump model—can be said to lie between the nearest-neighbor settings of $\mathbb{Z}$ and $\mathbb{Z}^d$. Globally, these models are still able to exhibit one-dimensional behavior, but locally, they behave more like random walks on a general graph. In these settings, directional transience and ballisticity have been given various characterizations, most in terms of Lyapunov exponents of infinite products of random matrices (see, for example, [16],[17],[18],[19], [20]). These exponents cannot in general be computed exactly, although some of them can be well approximated. Moreover, the characterizations of ballisticity have relied on various strong ellipticity assumptions, which preclude the Dirichlet model.

Although RWDE have proven to be a fruitful model for nearest-neighbor RWRE on $\mathbb{Z}^d$, we know of only one paper directly studying RWDE on $\mathbb{Z}$ with bounded jumps or on strips. That paper [21] was the first to treat RWDE in *any* setting. It uses a connection between RWDE and directed edge reinforced random walks to provide a characterization of directional transience for the latter in the mold of [16]. That paper preceded the development of helpful tools that have since been applied to the analysis of nearest-neighbor RWDE on $\mathbb{Z}^d$.

Since then, some of the results on higher-dimensional RWDE obtained by Sabot, Tournier, and Bouchet have not used a nearest-neighbor assumption, and therefore apply to all i.i.d.

RWDE with bounded jumps, including those on $\mathbb{Z}$. For example, it is shown in [22] that for RWDE where the annealed expectation of the first step is nonzero, the walk is almost-surely transient in the direction of that expectation. A remark in that paper points out that the proof does not use a nearest-neighbor assumption, and therefore the result applies to all RWDE with bounded jumps.

However, significant gaps remain between what is known for nearest-neighbor RWDE and RWDE with bounded jumps. In two dimensions, even a 0-1 law for directional transience (which has been proven for all nearest-neighbor RWRE on $\mathbb{Z}^2$ [23], on all bounded-jump RWDE on $\mathbb{Z}^d$, $d \geq 3$ [14], and on all bounded-jump RWRE on $\mathbb{Z}$ [16]) has not been proven for RWDE with bounded jumps. A comprehensive study of ballisticity of RWDE has only been done for dimensions 3 and higher [13], so in one dimension with bounded jumps, nothing about ballisticity, limiting distributions, large deviations, etc. has been shown for the Dirichlet regime beyond what is known for general RWRE on $\mathbb{Z}$ with bounded jumps (and in fact many of the results proven for RWRE on $\mathbb{Z}$ with bounded jumps require some version of a uniform ellipticity assumption that excludes Dirichlet environments).

1.2 Thesis structure

This thesis fills the aforementioned gaps as regards directional transience and ballisticity. It therefore has three primary goals:

1. To prove a 0-1 law for directional transience on $\mathbb{Z}^2$ with bounded jumps (which we do for general i.i.d. environments, not just Dirichlet environments);

2. To complete a characterization of directional transience in a given direction for RWDE with bounded jumps on $\mathbb{Z}^d$ by demonstrating recurrence in the case where the annealed expectation of the first step is zero; and

3. To characterize ballisticity for all RWDE on $\mathbb{Z}$ in terms of the Dirichlet parameters.

The rest of this chapter formally introduces the model and notation. Chapter 2 states our main results. Chapter 3 accomplishes the first two of the above goals, and Chapter 4 accomplishes the third. Chapter 4 also provides criteria for ballisticity that apply to all i.i.d.

RWRE with bounded jumps on $\mathbb{Z}$, not just Dirichlet ones, and proves some additional results that allow us to better understand the various phenomena affecting ballisticity. Chapter 5 continues this deeper exploration of the phenomena affecting ballisticity by proving some results about accelerated, continuous-time RWDE, in the mode of [14]. Chapter 6 gathers open questions presented throughout the thesis and poses an additional one. This thesis is based on two papers which the author has submitted for publication and posted to the arxiv. Most of Chapter 3 is based on [24], and most of Chapter 4 is based on [25]. Chapter 5 is based on material that was written for inclusion in [25] but was ultimately omitted for the sake of space.

1.3 Model and notation

We formally describe our model. Because there are many definitions and symbols introduced throughout this paper, and introduced at various points, we provide Appendix C to help the reader keep track of notation that is introduced here and elsewhere in the paper.

1.3.1 Random walks in random environments

Let V be a finite or countable set, and let $\Omega_V = \prod_{x\in V}\mathcal{M}_1(V)$, where $\mathcal{M}_1(V)$ is the set of probability measures on V, endowed with the topology of weak convergence. An *environment on* V is an element $\omega \in \Omega_V$, which can be thought of as a function from $V \times V$ to $[0, 1]$, with $\sum_{y\in V}\omega(x, y) = 1$ for all x. In the case where $V = \mathbb{Z}^d$, for each $x \in \mathbb{Z}^d$ we let $\omega^x \in \mathcal{M}_1(\mathbb{Z}^d)$ be the probability measure on $\mathbb{Z}^d$ given by $\omega^x(y) = \omega(x, x + y)$. Thus, $\omega = (\omega^x)_{x\in\mathbb{Z}^d}$ is the generic element of $\Omega_{\mathbb{Z}^d}$. For a given environment ω and $x \in V$, we can define P^x_ω to be the measure on $V^{\mathbb{N}_0}$ (with the natural sigma field) giving the law of a Markov chain $(X_n)_{n=0}^\infty$ started at x with transition probabilities given by ω. That is, $P^x_\omega(X_0 = x) = 1$, and for $n \geq 1$, $y \in V$, $P^x_\omega(X_{n+1} = y|X_0, \dots, X_n) = \omega(X_n, y)$.

Let $\mathcal{F}_V$ be the Borel sigma field on Ω_V (with respect to the product topology), and let P be a probability measure on $(\Omega_V, \mathcal{F}_V)$ (we often leave the $\mathcal{F}_V$ implicit and say P is a probability measure on Ω_V). For a given $x \in V$, we let $\mathbb{P}^x = P \times P_\omega^x$ be the measure on $\Omega_V \times V^{\mathbb{N}_0}$ induced by both P and P_ω. That is, for measurable events $A \subset \Omega_V, B \subset V^{\mathbb{N}_0}$,

$$\mathbb{P}^x(A \times B) = \int_A P_\omega^x(B) P(d\omega)$$

In particular, $\mathbb{P}^x(\Omega_V \times B) = E\left[P_\omega^x(B)\right]$. For convenience, we commit a small abuse of notation by using $\mathbb{P}^x$ to refer both to the measure we've described on $\Omega_V \times V^{\mathbb{N}_0}$ and also to its marginal $\mathbb{P}^x(\Omega_V \times \cdot)$ on $V^{\mathbb{N}_0}$. We call a measure P_ω^x on $V^{\mathbb{N}_0}$ a *quenched measure* of a random walk in random environment on V started at x, and we call the measure $\mathbb{P}^x$ the *annealed measure*. We will usually have $V = \mathbb{Z}^d$ for our main results, but certain intermediate results and arguments require different state spaces.

Most of the results in this paper concern measures P that are products of the measure of a Dirichlet distribution. However, some results are more general, and for these, we consider walks that satisfy the following conditions:

(C1) Under P, the transition probability vectors $(\omega^x)_{x \in \mathbb{Z}^d}$ are i.i.d.;

(C2) With P-probability 1, the Markov chain induced by ω is irreducible;

(C3) There is an $R > 0$ such that with P-probability 1, $\omega(x, y) = 0$ whenever $|x - y| > R$.

1.3.2 Random walks in Dirichlet environments

Let $\mathcal{H} = (V, E, w)$ be a weighted directed graph with vertex set V, edge set[1] $E \subseteq V \times V$, and a weight function $w : E \to \mathbb{R}^{>0}$. If $e = (x, y) \in E$, we say that e is an edge from x to y, and we say the *head* of e is $\overline{e} = y$ and the *tail* of e is $\underline{e} = x$. We say a set $S \subset V$ is *strongly connected* if for all $x, y \in S$, there is a path from x to y in $\mathcal{H}$ using only vertices in S. To the

[1]↑We define weighted directed graphs in a way that precludes multiple edges from sharing the same head and tail. However, we could expand our definition to include weighted directed multigraphs, and natural generalizations of the results that are true for graphs as we define them would still hold. Describing these generalizations would cause some notational inconvenience that is unnecessary for our purposes, such as defining random walks that keep track of edges taken as well as vertices visited. Nevertheless, we use multigraphs in some illustrations as a visual aid. Multiple edges from one vertex to another in our illustrations can be interpreted as a single edge whose weight is the sum of the weights of the edges depicted. By the amalgamation property reviewed in Section 1.4, identifying or splitting these edges that share the same head and tail does not affect the distributions of transition probability vectors between sites.

weighted directed graph $\mathcal{H}$, we can associate the Dirichlet measure $P_\mathcal{H}$ on $(\Omega_V, \mathcal{F}_V)$, which we now describe.

Recall the definition of the Dirichlet distribution: for a finite set I, take parameters $\alpha = (\alpha_i)_{i\in I}$, with $\alpha_i > 0$ for all i. The Dirichlet distribution with these parameters is a probability distribution on the simplex $\Delta_I := \{(x_i)_{i\in I} : \sum_{i\in I} x_i = 1\}$ with density

$$D\left((x_i)_{i\in I}\right) = C(\alpha)\prod_{i\in I} x_i^{\alpha_i - 1},$$

where $C(\alpha)$ is a normalizing constant. For the density with respect to the uniform measure on Δ_I, the normalizing constant is

$$C(\alpha) = \frac{\Gamma(\sum_{i\in I}\alpha_i)}{\Gamma(|I|)\prod_{i\in I}\Gamma(\alpha_i)}.$$

Define $P_\mathcal{H}$ to be the measure on Ω_V under which transition probabilities at the various vertices $x \in V$ are independent, and for each vertex $x \in V$, $(\omega(x, \overline{e}))_{\underline{e}=x}$ is distributed according to a Dirichlet distribution with parameters $(w(e))_{\underline{e}=x}$. With $P_\mathcal{H}$-probability 1, $\omega(x, y) > 0$ if and only if $(x, y) \in E$ for all $x, y \in V$. We will call a random environment chosen according to $P_\mathcal{H}$ a *Dirichlet environment on* $\mathcal{H}$. We will use $E_\mathcal{H}$ to denote the associated expectation, and $\mathbb{P}^x_\mathcal{H}$ and $\mathbb{E}^x_\mathcal{H}$ to denote the corresponding annealed measure and expectation.

1.3.3 Our main model

For the main model of this thesis, let $\mathcal{N}$ be a finite subset of $\mathbb{Z}^d$ that spans $\mathbb{Z}^d$ in the sense that $\sum_{i=1}^{\infty}(\mathcal{N} \cup \{0\}) = \mathbb{Z}^d$, and let $(\alpha_y)_{y\in\mathcal{N}}$ be positive weights. Let $\mathcal{G} = (\mathbb{Z}^d, E, w)$ be the weighted directed graph with vertex set $\mathbb{Z}^d$, edge set $E := \{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : y - x \in \mathcal{N}\}$, and weight function w with $w(x, y) = \alpha_{y-x}$ for all $(x, y) \in E$. Then $P_\mathcal{G}$ is the law of a Dirichlet environment on $\mathbb{Z}^d$ satisfying (C1), (C2), and (C3), and $\mathbb{P}^0_\mathcal{G}$ is the corresponding annealed measure for a walk started at 0.

1.3.4 Notation

When discussing RWRE on $\mathbb{Z}$, we will use interval notation to denote sets of consecutive integers rather than subsets of $\mathbb{R}$. Thus, for example, we will use $[1, \infty)$ to denote the set of integers to the right of 0. However, we make one exception, using $[0, 1]$ to denote the set of all real numbers from 0 to 1.

Let S^{d-1} be the unit sphere in $\mathbb{R}^d$. For a direction $\ell \in S^{d-1}$, $a \in \mathbb{R}$, $\diamond \in \{<, \leq, >, \geq\}$, and $\mathbf{X}$ a (finite or infinite) sequence in $\mathbb{Z}^d$, we define the stopping times

$$T^{\ell}_{\diamond a} = T^{\ell}_{\diamond a}(\mathbf{X}) := \inf\{n \geq 0 : (X_n \cdot \ell) \diamond a\}.$$

Similarly, for a point $x \in V$ or a set $S \subset V$, define

$$T_x = T_x(\mathbf{X}) := \inf\{n \geq 0 : X_n = x\}$$

and

$$T_S = T_S(\mathbf{X}) := \inf\{n \geq 0 : X_n \in S\}.$$

We often suppress the arguments $\mathbf{X}$ when the sequence intended is clear from the context. Likewise, we suppress the ℓ in the directional hitting times when the direction is clear from context. In particular, when $d = 1$, we always suppress the ℓ and assume $\ell = 1$, so that $X_n \cdot \ell = X_n$.

Finally, for any stopping time defined as an the first $n \geq 0$ such that satisfying a certain condition, we use the same notation but with a tilde ($\sim$) over it to denote the corresponding positive stopping time: that is, the first $n > 0$ satisfying the same condition.

1.4 Known results on Dirichlet environments

For a more comprehensive overview of random walks in Dirichlet environments, their properties, known results, and techniques used to achieve those results, see [15]. Here, we review specific results that will be useful to us.

1.4.1 Basic properties

We begin with basic properties of Dirichlet distributions.

Property (Amalgamation)**.** Assume $(U_i)_{i\in I}$ has Dirichlet distribution on Δ_I with parameters $(a_i)_{i\in I}$. Let $I_1, \ldots, I_r$ be a partition of I. The random vector $\left(\sum_{i\in I_k} U_i\right)_{1\le k\le r}$ on the simplex $\{(x_i)_{i=1}^r : \sum_{i=1}^r x_i = 1\}$ follows the Dirichlet distribution with parameters $\left(\sum_{i\in I_k} a_i\right)_{1\le k\le r}$.

Property (Restriction)**.** Assume $(U_i)_{i\in I}$ has Dirichlet distribution on Δ_I with parameters $(a_i)_{i\in I}$. Let J be a nonempty subset of I. The random vector $\left(\frac{U_i}{\sum_{j\in J} U_j}\right)_{i\in J}$, which takes values on the simplex Δ_J, follows the Dirichlet distribution with parameters $(a_i)_{i\in J}$ and is independent of $\sum_{j\in J} U_j$. It is also independent of $(U_k)_{k\notin J}$.

By the amalgamation property, the marginal distribution of each coordinate of a Dirichlet random vector is a beta distribution. The next property bounds the probability that a beta random variable is small.

Property (Moments)**.** Assume X is a beta random variable with parameters (a, b). Then there exists constants $0 < c < C$ such that for all $\varepsilon \in [0, 1]$,

$$c\varepsilon^a \le P(X < \varepsilon) \le C\varepsilon^a. \tag{1.1}$$

In particular, $E\left[\frac{1}{X^s}\right] < \infty$ if and only if $s < a$.

The amalgamation and restriction properties are also stated in [15], but the last sentence, "It is also independent of $(U_k)_{k\notin J}$," is not contained in the statement there. However, it follows from the rest of the restriction property. Let $I = \{1, \ldots, n\}$, and let $J = \{1, \ldots, m\}$. Then since $\left(\frac{U_i}{\sum_{j=1}^{m+1} U_j}\right)_{i=1}^{m+1}$ follows a Dirichlet distribution, we can apply the restriction property to get

$$\left(\frac{\frac{U_i}{\sum_{j=1}^{m+1} U_j}}{\sum_{r=1}^{m} \frac{U_r}{\sum_{j=1}^{m+1} U_j}}\right)_{i=1}^{m} \perp\!\!\!\perp \sum_{r=1}^{m} \frac{U_r}{\sum_{j=1}^{m+1} U_j},$$

from which it follows that

$$\left(\frac{U_i}{\sum_{r=1}^{m} U_r}\right)_{i=1}^{m} \perp\!\!\!\perp \frac{U_{m+1}}{\sum_{j=1}^{m+1} U_j}. \tag{1.2}$$

By a similar argument, we get

$$\left(\frac{U_i}{\sum_{r=1}^{m+1} U_r}\right)_{i=1}^{m+1} \perp\!\!\!\perp \frac{U_{m+2}}{\sum_{j=1}^{m+2} U_j}. \tag{1.3}$$

But since both sides of (1.2) are determined by the left side of (1.3), it follows that

$$\left(\frac{U_i}{\sum_{r=1}^{m} U_r}\right)_{i=1}^{m}, \quad \frac{U_{m+1}}{\sum_{j=1}^{m+1} U_j}, \quad \text{and} \quad \frac{U_{m+2}}{\sum_{j=1}^{m+2} U_j}$$

are all independent. Continuing the argument in the same manner, we see that all of the following are independent:

$$\left(\frac{U_i}{\sum_{r=1}^{m} U_r}\right)_{i=1}^{m}, \quad \frac{U_{m+1}}{\sum_{j=1}^{m+1} U_j}, \quad \frac{U_{m+2}}{\sum_{j=1}^{m+2} U_j}, \quad \dots, \quad \frac{U_{n-1}}{\sum_{j=1}^{n-1} U_j}, \quad U_n. \tag{1.4}$$

Now $(U_k)_{k=m+1}^{n}$ is a function of the last $n-m$ terms of (1.4) (that is, all terms but the first), and so it is independent of the first.

1.4.2 Edge reinforcement

Dirichlet environments were first studied for their connection to a stochastic process called a *directed edge reinforced random walk* (DERRW), which we now define. For a weighted directed graph $\mathcal{H} = (V, E, w)$, and for an initial vertex $x_0 \in V$, we define the stochastic processes $(X_n)_{n=0}^{\infty}$ and $(r(e,n))_{e\in E, n\geq 0}$ as follows: with probability 1, $X_0 = x_0$ and $r(e,0) = w(e)$ for all e. If $X_n = x$ and e_1 is an edge with $\underline{e_1} = x$ (i.e., the edge is rooted at x), then the walk takes edge e_1 (so that $X_{n+1} = \overline{e_1}$) with probability $\frac{r(e_1,n)}{\sum_{\underline{e}=x} r(e,n)}$. Each time an edge is taken, its weight r is increased by 1; otherwise, weights do not change. That is, $r(e, n+1) = \begin{cases} r(e,n)+1 & \text{if } (X_n, X_{n+1}) = e \\ r(e,n) & \text{otherwise} \end{cases}$. The process $(X_n)_{n=0}^{\infty}$ is the DERRW on $\mathcal{H}$ started at x_0. The following lemma was first shown in [26] and [21]. It can be shown using the fact that asymptotic proportions of colors in a Pólya urn follows a Dirichlet distribution, together with de Finetti's theorem.

Lemma 1.4.1 ([15], Lemma 2)**.** *Let V be a set, and let $\mathcal{H} = (V, E, w)$ be a weighted directed graph with vertex set V. Then the law of a DERRW on $\mathcal{H}$ started at vertex $x \in V$ is the annealed law $\mathbb{P}^x_{\mathcal{H}}$ of the RWDE on $\mathcal{H}$.*

1.4.3 Time reversal lemma

We now describe an important time-reversal lemma for Dirichlet environments. Let $\mathcal{H} = (V, E, w)$ be a weighted directed graph, and let $P_{\mathcal{H}}$ be the associated product Dirichlet measure on the set Ω_V of environments on V. For a vertex $x \in V$, the *divergence* of x in $\mathcal{H}$ is $\operatorname{div}(x) = \sum_{\overline{e}=x} w(e) - \sum_{\underline{e}=x} w(e)$. If the divergence is zero for all x, we say the graph $\mathcal{H}$ has zero divergence. Now if $\mathcal{H}$ is a finite, strongly connected graph, then for $P_{\mathcal{H}}$–a.e. $\omega \in \Omega_V$, there exists an invariant probability for the Markov chain corresponding to ω. Call this invariant measure π^ω. Define the *time reversed* environment $\check{\omega}$ by

$$\check{\omega}(x, y) := \frac{\pi^\omega(y)}{\pi^\omega(x)} \omega(y, x)$$

We can check that $\check{\omega}$ is an environment by noting

$$\sum_{y \in V} \check{\omega}(x, y) = \frac{1}{\pi^\omega(x)} \sum_{y \in V} \pi^\omega(y) \omega(y, x) = 1.$$

Moreover, the probability, under ω, of taking any loop is equal to the probability, under $\check{\omega}$, of taking the reversed loop. To see this, note that if $v_n = v_0$, then

$$\begin{aligned}
P^{v_0}_{\check{\omega}}(X_1 = v_{n-1}, X_2 = v_{n-2}, \ldots, X_n = v_0) &= \prod_{i=0}^{n-1} \check{\omega}(v_{i+1}, v_i) \\
&= \prod_{i=0}^{n-1} \frac{\pi^\omega(v_i)}{\pi^\omega(v_{i+1})} \omega(v_i, v_{i+1}) \\
&= \frac{\prod_{i=0}^{n-1} \pi^\omega(v_i)}{\prod_{i=0}^{n-1} \pi^\omega(v_{i+1})} \prod_{i=0}^{n-1} \omega(v_i, v_{i+1}) \\
&= \prod_{i=0}^{n-1} \omega(v_i, v_{i+1}) \\
&= P^{v_0}_{\omega}(X_1 = v_1, X_2 = v_2, \ldots, X_n = v_n).
\end{aligned}$$

The following lemma can be proven using Lemma 1.4.1. It was first proven analytically in [27], and its probabilistic proof was first given in [28]. Let $\check{\mathcal{H}}$ be the graph made by reversing all edges of $\mathcal{H}$ and keeping the same weights[2], and let $P_{\check{\mathcal{H}}}$ be the associated measure on Ω_V.

Lemma 1.4.2 ([15], Lemma 3.1)**.** *If the graph $\mathcal{H}$ has zero divergence, then the law of $\check{\omega}$ is $P_{\check{\mathcal{H}}}$.*

In other words, drawing an environment ω according to $P_{\mathcal{H}}$ and then time-reversing it is the same as reversing the edges of $\mathcal{H}$ to get $\check{\mathcal{H}}$ and then drawing an environment according to $P_{\check{\mathcal{H}}}$. This lemma implies that the probability, under $\mathbb{P}_{\mathcal{H}}$, of taking any loop is equal to the probability, under $\mathbb{P}_{\check{\mathcal{H}}}$, of taking the reversed loop. Indeed, for our purposes, the use of Lemma 1.4.2 comes from the following corollary.

Corollary 1.4.3. *Let $\mathcal{H}$ be as described above, and let $x, y \in V$ be such that there is an edge e from y to x in $\mathcal{H}$. Then, letting $\tilde{T}_x$ denote the first positive hitting time of x,*

1. *The law of $P^x_\omega(X_{\tilde{T}_x - 1} = y)$ under $P_{\mathcal{H}}$ is the law of $P^x_\omega(X_1 = y) = \omega(x, y)$ under $P_{\check{\mathcal{H}}}$.*
2. $\mathbb{P}^x_{\mathcal{H}}(X_{\tilde{T}_x - 1} = y) = \mathbb{P}^x_{\check{\mathcal{H}}}(X_1 = y) = \frac{w(y,x)}{\sum_{v \in V} w(v,x)}$.

The formula for the probability as a fraction comes from either Lemma 1.4.1 or the amalgamation property and the fact that the expectation of a beta random variable with parameters (a, b) is $\frac{a}{a+b}$.

1.4.4 Moments of quenched Green functions

The next lemma we recall was proven by Tournier [29]. We will refer to it as Tournier's lemma. To formally state this lemma, we need some notation. Let $\mathcal{H} = (V, E, w)$ be a weighted directed graph. For a walk $\mathbf{X} = (X_n)_{n=0}^\infty$ on V with $x \in V$, $N_x(\mathbf{X}) = \#\{n \in \mathbb{N}_0 : X_n = x\}$ is the number of times the walk is at site x. We usually write it as N_x if we are able to do so without ambiguity. For a subset $S \subset V$, let $N_S = \sum_{x \in S} N_x$. Also, for a set $S \subseteq V$, define

$$\beta_S := \sum_{\underline{e} \in S,\ \overline{e} \notin S} w(e). \tag{1.5}$$

[2]Formally, if $e = (a, b)$, then let $\check{e} = (b, a)$. Then define $\check{w}(\check{e}) = w(e)$ and $\check{E} = \{\check{e} : e \in E\}$. Now let $\check{\mathcal{H}} = (V, \check{E}, \check{w})$.

This parameter β_S is the sum of the weights of all edges exiting the set S.

Lemma 1.4.4 (Tournier's lemma; see [29], Theorems 1 and 2)**.** *Let $\mathcal{H} = (V \cup \{\partial\}, E, w)$ be a finite weighted directed graph with ∂ a unique sink reachable from every other site. We denote by $P_{\mathcal{H}}$ the corresponding Dirichlet distribution on environments.*

For every $s > 0$, the following statements are equivalent:

1. $E_{\mathcal{H}}[E_\omega^x[N_x]^s] < \infty$.

2. *For every strongly connected subset S of V with $x \in S$, $\beta_S > s$.*

In particular, by letting $s = 1$, we see that $\mathbb{E}_{\mathcal{H}}^x[N_x] < \infty$ if and only if for every strongly connected subset S of V containing x, $\beta_S > 1$.

The formulation given in Theorem 1 of [29] is in terms of strongly connected sets of edges rather than vertices, but implies ours. Tournier's original formulation is as follows: For a weighted directed graph $\mathcal{H} = (V, E, w)$ and a subset $A \subset E$, let $\overline{A} = \{\overline{e} : e \in A\}$, $\underline{A} = \{\underline{e} : e \in A\}$, and $\underline{\overline{A}} = \overline{A} \cup \underline{A}$. Say A is *strongly connected* if any two vertices in $\underline{A}$ can communicate using only edges in A (note that this implies $\overline{A} = \underline{A} = \underline{\overline{A}}$). Now define $\beta_A = \sum_{\underline{e} \in \underline{A}} w(e) \mathbb{1}_{\{e \notin A\}}$.

Lemma 1.4.5 ([29], Theorem 1)**.** *Let $\mathcal{H} = (V \cup \{\partial\}, E, w)$ be a finite weighted directed graph with ∂ a unique sink reachable from every other site. Denote by $P_{\mathcal{H}}$ the corresponding Dirichlet distribution on environments. For every $s > 0$, the following statements are equivalent:*

1. $E_{\mathcal{H}}[E_\omega^x[N_x]^s] < \infty$.

2. *For every strongly connected subset A of E such that $x \in \underline{\overline{A}}$, $\beta_A > s$.*

Every β_S for a set S of vertices is β_A for the set A of edges between vertices in S. On the other hand, for any strongly connected set A of edges, one can take S to be the set of heads or tails of edges in A and take A' to be the set of edges between vertices in S (so that $A \subset A'$). Then $\beta_S = \beta_{A'} < \beta_A$. From this, one can check that Tournier's formulation implies ours.

In turn, Tournier's original version can be deduced from ours by considering a modified graph $\mathcal{H}'$ with "an extra vertex added in the middle of each edge"; that is, if e is an edge from x to y in $\mathcal{H}$ with weight $w(e)$, then $\mathcal{H}'$ has an extra vertex z_e, and instead of an edge from x to y, there is an edge from x to z_e with weight $w(e)$ and an edge from z_e to y with weight 1 (this weight is arbitrary, as it is the only edge exiting z_e). Thus, every strongly connected set A of edges in $\mathcal{H}$ corresponds to a strongly connected set $S(A)$ of vertices in $\mathcal{H}'$, with $\beta_{S(A)} = \beta_A$. Moreover, for any $x \in V$, drawing a path on $\mathcal{H}'$ according to $\mathbb{P}^x_{\mathcal{H}'}$ and then deleting the extra vertices gives a path on $\mathcal{H}$ drawn according to $\mathbb{P}^x_{\mathcal{H}}$.

2. MAIN RESULTS

This chapter outlines the main results of the thesis, provides additional background on them, and gives an idea of the proofs.

2.1 0-1 law

A natural first direction of study for RWRE on $\mathbb{Z}^d$ is characterizing directional transience. For $\ell \in S^{d-1}$, define

$$A_\ell := \{\mathbf{X} \in (\mathbb{Z}^d)^{\mathbb{N}_0} : \lim_{n\to\infty} X_n \cdot \ell = \infty\}.$$

In 1981, Kalikow [30] asked whether, for i.i.d. RWRE in 2 dimensions, the x-coordinate of the walker's position must approach infinity with probability either 0 or 1. He was able to show that the walk hits the y-axis infinitely often with probability either 0 or 1. In other words, if ℓ is a unit vector in the x direction, Kalikow showed $\mathbb{P}^0(A_\ell \cup A_{-\ell}) \in \{0,1\}$ and asked whether it can be shown that $\mathbb{P}^0(A_\ell) \in \{0,1\}$. In 2001, Zerner and Merkl [23] answered this question in the affirmative for nearest-neighbor, i.i.d., elliptic RWRE in 2 dimensions, and not just for the x direction but for any direction $\ell \in S^1$. They also showed that the i.i.d. assumption is necessary by providing a non-i.i.d. counterexample where the 0-1 law fails. A 0-1 law for directional transience of nearest-neighbor, i.i.d. RWRE in dimensions $d \geq 3$ is still a major open conjecture. We extend the result of Zerner and Merkl by removing the nearest-neighbor assumption, showing that for i.i.d. elliptic RWRE with bounded jumps on $\mathbb{Z}^2$, the 0-1 law holds for all directions $\ell \in S^1$. Our proof is largely based on that of [31], which is a simplification of the proof given in [23]. However, the removal of the nearest-neighbor assumption creates a need for some additional work.

Theorem 2.1.1. *Let $d = 2$, and let assumptions (C1), (C2), and (C3) hold, and let $\ell \in S^1$. Then $\mathbb{P}^0(A_\ell) \in \{0,1\}$, where A_ℓ is the event $\lim_{n\to\infty} X_n \cdot \ell = \infty$.*

Jump to proof.

The idea behind Zerner's proof in [31] is that if the probability of transience in both direction ℓ and direction $-\ell$ is positive, then with non-vanishing probability, one should be able to start two walks in the same environment on different sides of a wide strip and have both walks cross the strip and exit on the opposite side from where they started (call this the *strip traversal event*). If the starting points are chosen correctly, this should lead to the paths of the walks intersecting at least half the time. But if the paths intersect, it is at a point that at least one of the walks has traveled a long distance to reach. A walk started on the right side of a strip (thinking of ℓ as "to the right") that has traveled a great distance to the left ought to have reached a point in the environment where there is a very high probability of being transient to the left. However, on the event in question, there is also a path started from the left side of the strip that goes through that same point and then travels all the way to the right side of the strip. Thus, if the two walks both cross the strip and their paths intersect, then an event of vanishingly small probability must occur. But if the paths intersect at least half the time the strip traveral event occurs, then the probability of the strip traversal event must vanish as the strip gets wider and wider. This implies that the probability of transience in one direction or the other must be zero.

The difficulty in adapting this proof to the case of bounded jumps is that it become possible for the paths of walks to "cross" (in the sense that the continuous linear interpolations of their paths cross) without actually sharing a vertex. One can, however, use the boundedness of jumps to show that in this case, the paths must come within a bounded distance of each other. Showing that they come within a bounded distance of each other at least half the time requires dealing with a couple of fringe cases that do not show up under a nearest-neighbor assumption. But the more significant challenge is to use the proximity of the two paths to get the same result as in Zerner's argument. To do this, we use the fact that if one walk comes close enough to the path of the other, its annealed probability of actually landing on a vertex of that path is bounded from below. Unlike in Zerner's argument, we are not able to get any sort of a bound on the probability that the strip traversal event occurs *and* the paths intersect at some vertex. But we are able to compare the probability of the strip traversal event to the probability that one of the walks crosses a strip and exits on the opposite side, and that the other walk travels a long distance before intersecting the path of the first walk. This event has similar properties to the strip traversal event with intersection, and we are able to show that its probability vanishes.

2.2 Directional recurrence of balanced RWDE on $\mathbb{Z}^d$

We give an application of our 0-1 law to RWDE. For a given direction $\ell \in S^{d-1}$, the question of transience and recurrence in direction ℓ is completely understood for nearest-neighbor RWDE, due to [28], [14], and [22]. Tournier remarks in [22] that many of the results used in the characterization of transience do not rely on the nearest-neighbor assumption, so much of what was known in the nearest-neighbor case carries over to the bounded-jumps case.

However, not everything carries over directly. One crucial step toward characterizing directional transience is a 0-1 law. As Tournier points out in his aforementioned remark, the proof of the 0-1 law for RWDE in dimensions $d \geq 3$ given in [14] does not require the nearest-neighbor assumption, but the proof for RWRE in dimension $d = 2$ in [23] and [31] does require the nearest-neighbor assumption. Our extension of the 0-1 law for $d = 2$ to bounded jumps means that for RWDE with bounded jumps, the 0-1 law is now proven for all dimensions.

Removing the nearest-neighbor assumption creates one other obstacle to fully characterizing directional transience in a given direction. When the annealed drift is zero, the proof of directional recurrence in the nearest-neighbor argument relies on a symmetry that does not necessarily exist in the bounded-jump case. Additional work is therefore needed to prove that zero drift implies recurrence in any direction.

The directional recurrence/transience result known for nearest-neighbor RWDE (i.e. when $\mathcal{N}$ is the set of nearest neighbors of 0) states that for a given direction $\ell \in S^{d-1}$, transience and recurrence in direction ℓ under $\mathbb{P}^0_{\mathcal{G}}$ are characterized by the relationship between ℓ and the annealed drift.

Theorem ([15, Theorem 1]). *Let $\mathbb{P}^0_{\mathcal{G}}$ be the measure of a nearest-neighbor RWDE on $\mathbb{Z}^d$. Let $\Delta = \mathbb{E}^0[X_1]$ be the annealed drift, and let $\ell \in S^{d-1}$. Then $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 1$ if and only if $\ell \cdot \Delta > 0$; otherwise, $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 0$.*

We extend this theorem to RWDE with bounded jumps.

Theorem 2.2.1. *Let $\mathbb{P}^0_{\mathcal{G}}$ be the measure of a RWDE with bounded jumps on $\mathbb{Z}^d$. Let $\Delta = \mathbb{E}^0[X_1]$ be the annealed drift, and let $\ell \in S^{d-1}$. Then $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 1$ if and only if $\ell \cdot \Delta > 0$; otherwise, $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 0$.*

Jump to proof.

Based on Tournier's remark in [22] (pointing out which arguments used in the proof of the theorem from [15] do not rely on the nearest-neighbor assumption), Theorem 2.2.1 is known to be true provided $\Delta \neq 0$ and $d \neq 2$.

If $\Delta \neq 0$ and $d = 2$, we know from [22] that $\ell \cdot \Delta > 0$ implies $\mathbb{P}^0_{\mathcal{G}}(A_\ell) > 0$, and from [12, Theorem 1.8] that $\ell \cdot \Delta = 0$ implies $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 0$ (the arguments in [12] are given for the nearest-neighbor case, but can be easily modified to work for environments satisfying (C1), (C2), and (C3)). From here, our 0-1 law of Theorem 2.1.1 allows us to reach the conclusion of Theorem 2.2.1.

The only remaining case is where $\Delta = 0$. In the nearest-neighbor case, $\Delta = 0$ implies a symmetry that forces $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = \mathbb{P}^0_{\mathcal{G}}(A_{-\ell})$ for all directions ℓ. The 0-1 laws of [23] for $d = 2$ and of [14] for $d \geq 3$ then yield the conclusion $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 0$ for all ℓ. In the bounded-jumps case, zero drift does not imply symmetry, so even the 0-1 law of Theorem 2.1.1 is not by itself enough to prove the theorem. Theorem 2.2.1 will be proven if we can prove the following theorem, which will rely on Theorem 2.1.1 for the case $d = 2$.

Theorem 2.2.2. *Let $\mathbb{P}^0_{\mathcal{G}}$ be the measure of a RWDE with bounded jumps on $\mathbb{Z}^d$. If $\Delta = 0$, then $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 0$ for all $\ell \in S^{d-1}$.*

Jump to proof.

As is common for the proofs of results in RWDE, our proof involves comparing the graph $\mathcal{G}$ to a sequence of larger and larger finite graphs $(\mathcal{H}_{N,L})$, which look like $\mathcal{G}$ except possibly near boundaries, and applying Corollary 1.4.3. The finite graphs we construct are similar to those constructed by Tournier in [22] for the characterization of transience in the nonzero-drift case.

For simplicity, we first prove the result for directions ℓ with rational slopes. Extending it to all ℓ is not as immediate as one might hope. Indeed, the following conjecture remains open for general RWRE.

Conjecture 2.2.3. *Let $\mathbb{P}^0$ be the law of an i.i.d. RWRE on $\mathbb{Z}^d$, and let S^{d-1} be the set of a unit vectors in $\mathbb{R}^d$. Then for all $\ell \in S^{d-1}$, if $\mathbb{P}^0(A_\ell) > 0$, then there exsts a neighborhood $U \subseteq S^{d-1}$ such that $\mathbb{P}^0(A_{\ell'}) > 0$ for all $\ell' \in U$.*

In the nearest-neighbor case of RWDE, Conjecture 2.2.3 is seen to be true from [15, Theorem 1], and in the bounded-jump case it will follow from Theorem 2.2.1, once it is proven (again, it only remains to prove Theorem 2.2.2).

The idea behind our proof for rational directions is to define a graph $\mathcal{H}_{N,L}$ using a large slab from $\mathbb{Z}^d$ with periodic boundary conditions, and to add a left endpoint ∂ and right endpoint M. Edges entering and exiting the slab on the left are interpreted as edges to and from ∂, while edges entering and exiting the slab on the right are interpreted as edges to and from M. We also add special edges between ∂ and M. The graph will be described more formally in the actual proof, but we give a picture of it here, in Figure 2.1.

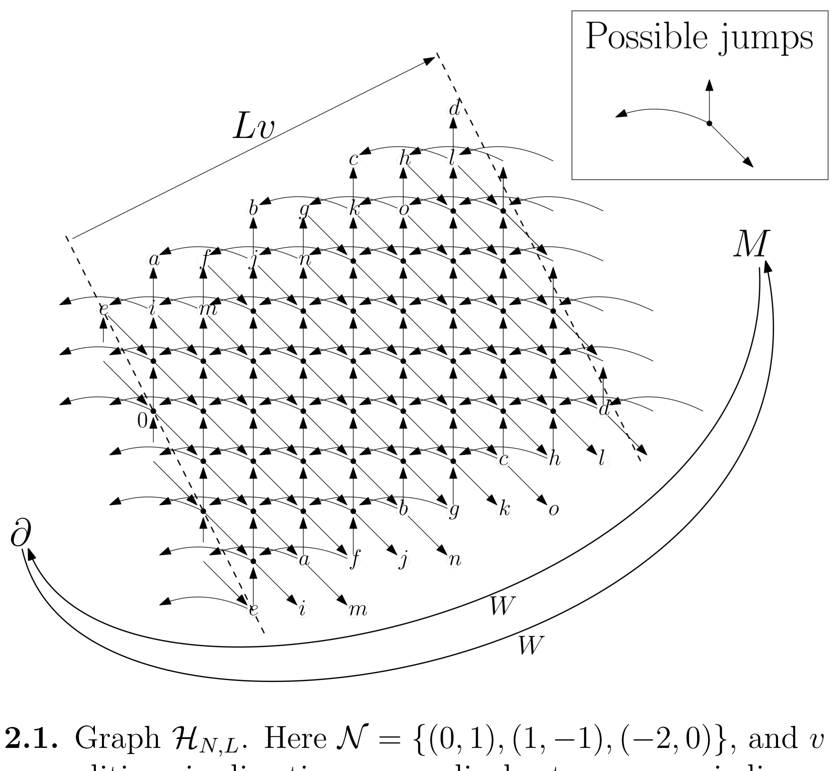

Figure 2.1. Graph $\mathcal{H}_{N,L}$. Here $\mathcal{N} = \{(0,1),(1,-1),(-2,0)\}$, and $v = (2,1)$. Boundary conditions in direction perpendicular to v are periodic; vertices labeled with the same letters are identified. Arrows to and from the main part of the graph on the left are understood to originate from or terminate at ∂, and similarly with M on the right side.

Due to the assumption that $\triangle = 0$, we are able to make this graph satisfy the zero divergence criterion provided the edges from ∂ to M and M to ∂ are given the same weight W. We choose this W so that a walk started at ∂ has a $\frac{1}{2}$ chance of stepping to M on its first step. By Corollary 1.4.3, the probability that the first return to 0 is by the special edge from M is also $\frac{1}{2}$. We assume for a contradiction that $\mathbb{P}^0_{\mathcal{G}}(A_{-\ell}) > 0$. By the 0-1 law of [16] for dimension $d = 1$, the 0-1 law of [14] for dimension $d \geq 3$, or Theorem 2.1.1 for dimension $d = 2$, this assumption implies $\mathbb{P}^0_{\mathcal{G}}(A_{-\ell}) = 1$. Using this fact and the fact that the main part of the graph $\mathcal{H}_{N,L}$ looks and feels like $\mathbb{Z}^d$ (taking care to ensure that the graph is large enough in directions perpendicular to ℓ that its periodic boundary conditions are vanishingly unlikely to be used), we are able to show that the probability of a walk from ∂ reaching M by the main part of the graph before returning to ∂ vanishes as the size of the graph increases. Thus, in the limit, the only way the first return to ∂ can be by the special edge from M is if the first step from ∂ takes the special edge to M. However, a positive probability of transience in direction $-\ell$ also implies that if the first step from ∂ takes the special edge to M, then there is a positive probability, bounded from below, that the walk returns to ∂ through the main part of the graph rather than taking the special edge. Thus, on the one hand, in order for the walk to first return to ∂ by the special edge from M, it must step to M on its first step (which happens with probability $\frac{1}{2}$), and then step from M to ∂ by the special edge before making its way back down through the main part of the graph (which happens with probability bounded away from 1). This implies that the probability that the first return to ∂ is by the special edge from M is eventually strictly less than $\frac{1}{2}$. On the other hand, we have by Corollary 1.4.3 that it is exactly $\frac{1}{2}$. This contradiction completes the proof.

To generalize to directions ℓ with irrational slopes, it is not enough to cite the result for rational slopes. Rather, we repeat the argument more carefully, this time taking, for each L a direction v with rational slopes, sufficiently close to ℓ (and approaching ℓ as L increases). We define graphs $\mathcal{H}_{N,L}$ in terms of these directions v. We show that appropriate upper and lower bounds on probabilities of the walk traveling through the main part of the graph from one endpoint to the other still hold, provided these rational directions v are close enough to ℓ, and provided the graphs are large enough in directions perpendicular to ℓ. We then use the properties of the graphs $\mathcal{H}_{N,L}$ and Corollary 1.4.3 to get the same contradiction as before.

2.3 Ballisticity of RWDE on $\mathbb{Z}$

We now consider the case $d = 1$. Then $\mathcal{N}$ is a finite subset of $\mathbb{Z}$ such that the GCD of all $i \in \mathcal{N}$ is 1. Let $L = \min(\mathcal{N})$ and $R = \max(\mathcal{N})$. We have $\alpha_i > 0$ for $i \in \mathcal{N}$. For this one-dimensional model, also define $\alpha_i := 0$ for any $i \notin \mathcal{N}$, so that $(\alpha_i)_{i=-L}^{R}$ are non-negative real numbers with $\alpha_i > 0$ if and only if $i \in \mathcal{N}$. We have a random walk in a Dirichlet random environment on $\mathbb{Z}$ with jumps to the left up to L steps and to the right up to R steps, with transition probability vectors given by i.i.d. Dirichlet random vectors with parameters $(\alpha_i)_{i\in\mathcal{N}}$. Our graph $\mathcal{G}$ has vertex set $\mathbb{Z}$, edge set $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y - x \in \mathcal{N}\}$, and weight function $(x, y) \mapsto \alpha_{y-x}$. An example with $L = R = 2$ is represented in Figure 2.2 (here, and in other illustrations of graphs, we depict the case where $\alpha_0 = 0$, but our model does allow for $\alpha_0 > 0$).

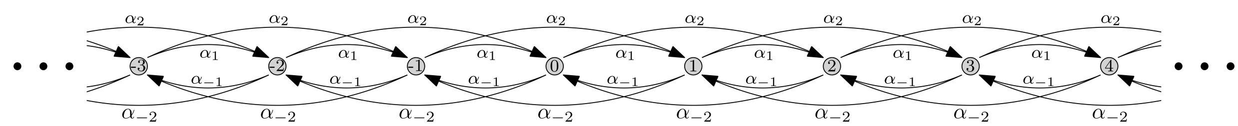

Figure 2.2. A portion of the graph $\mathcal{G}$ with $L = R = 2$.

Our main concern in Chapter 4 is to characterize ballisticity of a transient random walk in a Dirichlet environment on $\mathcal{G}$ started at 0 in terms of the α_i. From the irreducibility assumption, it follows that there is an m large enough that every interval of length m is strongly connected in $\mathcal{G}$. Let m_0 be such an integer, chosen large enough that also $m_0 \geq \max(L, R)$. We will use this m_0 in several proofs throughout this paper.

For i.i.d. RWRE on $\mathbb{Z}$ with bounded jumps, there necessarily exists a deterministic v with $\lim_{n\to\infty} \frac{X_n}{n} = v$ almost surely (see Appendix A). When the walk is recurrent, $v = 0$. We assume the walk is transient to the right, so that $v \geq 0$, and characterize the ballistic regime $v > 0$.

Our results on ballisticity comprise a sort of mixture of characterizations for the nearest-neighbor cases on $\mathbb{Z}$ and on $\mathbb{Z}^d$, $d \geq 3$. These characterizations are quite different from each other, as they reflect substantially different ways that a walk may "get stuck." (The case $d = 2$ is still open.) For $d \geq 3$, ballisticity has a simple characterization in terms of a parameter κ_0 (called κ in [14]). The fact that Dirichlet distributions are not uniformly elliptic allows environments to contain arbitrarily severe "traps" where a walk can get stuck for a long time. In fact, if enough parameters of the Dirichlet distribution are sufficiently small, there are finite subgraphs whose annealed expected exit times are infinite, causing zero limiting speed. It was shown in [29, Theorem 1] that the parameter κ_0, which represents the minimal amount of weight exiting a finite set, controls finite moments of the quenched expected exit times of finite traps containing the origin. In the case $d \geq 3$, a directionally transient walk is ballistic if and only if $\kappa_0 > 1$ [15, Theorem 5], reflecting the idea that finite traps are the only way directional transience with zero speed can occur in the case $d \geq 3$.

The nearest-neighbor case $d = 1$, where probabilities $\omega(x, x+1)$ of stepping to the right are given by beta random variables with parameters (α_1, α_{-1}), is different. Here, $\kappa_0 = \alpha_1 + \alpha_{-1}$, and this parameter still controls finite traps, but it is possible for a walk to have zero speed even if $\kappa_0 > 1$. In fact, ballisticity is controlled by another parameter, $\kappa_1 = \alpha_1 - \alpha_{-1}$, which also characterizes directional transience, and which is the unique positive number (studied in a more general setting by Kesten, Kozlov, and Spitzer in [32], and there called κ) such that $E\left[\left(\frac{1-\omega(0,1)}{\omega(0,1)}\right)^{\kappa_1}\right] = 1$. In the nearest-neighbor case $d = 1$, a walk is transient to the right if and only if $\kappa_1 > 0$, and in that case it is ballistic if and only if $\kappa_1 > 1$. These results come from a direct application of the characterizations of directional transience and ballisticity given in [1]. Here, as in higher dimensions, $\kappa_0 \leq 1$ is enough to cause finite trapping that would slow the walk down to zero speed. However, because we always have $\kappa_1 < \kappa_0$, the walk is already not ballistic in this case, and thus the value of κ_1 alone determines ballisticity.

The parameters κ_0 and κ_1 can be given definitions that apply to our model as well. The parameter κ_0 can be defined for RWDE on any weighted directed graph. For a weighted, directed graph $\mathcal{H} = (V, E, w)$ and vertex $x \in V$, one may define $\kappa_0 = \kappa_0(\mathcal{H}, x)$ as the minimal total weight of edges exiting a finite, strongly connected set of vertices containing x in $\mathcal{H}$.

The smaller $\kappa_0(\mathcal{H}, x)$ is, the greater is the propensity of a walk drawn according to $\mathbb{P}^x_{\mathcal{H}}$ to get stuck for a long time in a finite trap containing x [29]. We give a precise definition for our graph $\mathcal{G}$ in Section 4.2.1; see (4.11) (the definition is not vertex-dependent due to the translation-invariance of $\mathcal{G}$). We do not give an explicit formula for κ_0, which is defined as an infimum over an infinite set of sums, but we show that it is in fact a minimum over finitely many sums, and provide an algorithm to compute it directly.

For κ_1, let $d^+ = \sum_{i=1}^{R} i\alpha_i$ and $d^- = \sum_{i=-L}^{-1} |i|\alpha_i$. Then $\kappa_1 := d^+ - d^-$ is the weighted sum of the weights α_i, and its sign is the sign of $\mathbb{E}^0_{\mathcal{G}}[X_1]$. In fact, let c^+ and c^- be the unweighted sums $\sum_{i=1}^{R} \alpha_i$ and $\sum_{i=-L}^{-1} \alpha_i$, respectively. One can check using Lemma 1.4.1, or simply using the amalgamation property and expectation of a beta random variable, that $\mathbb{E}^0_{\mathcal{G}}[X_1] = \frac{\kappa_1}{c^- + \alpha_0 + c^+}$. Notice that when $L = R = 1$, κ_1 reduces to $\alpha_1 - \alpha_{-1}$ By Theorem 2.2.1, the walk is $\mathbb{P}^0_{\mathcal{G}}$-almost surely transient in the direction of κ_1 when it is not 0, and recurrent when $\kappa_1 = 0$. We will see that the parameter κ_1 plays a key role in characterizing ballisticity as well.

We show that unlike in the nearest-neighbor case $L = R = 1$, where we always have $\kappa_0 > \kappa_1$, our model allows the ordered pair (κ_0, κ_1) to take on any value in the first quadrant of $\mathbb{R}^2$; see Proposition B.0.2. Both κ_0 and κ_1 must be greater than 1 in order to achieve ballisticity. When $\kappa_0 \leq 1$, the walk has zero speed because of the relatively high likelihood of getting stuck in a region of bounded size for a long time. When $\kappa_1 \leq 1$, the walk has zero speed because of the relatively high likelihood of repeatedly backtracking over regions of all sizes. When both are greater than 1, the walk is ballistic.

The appearance of the dual possibilities of finite trapping and large-scale backtracking seems to be a new phenomenon in RWRE with bounded jumps. Previous characterizations of ballisticity use ellipticity assumptions strong enough to preclude finite trapping, and therefore do not cover cases where walks can get stuck in these two different ways.

2.3.1 General ballistic criteria

While our main results are for RWDE, part of the proof requires obtaining some results that apply to general RWRE on $\mathbb{Z}$ with bounded jumps. Section 4.1 provides two characterizations of ballisticity under conditions (C1), (C2), and (C3). It was shown in [16] that under these assumptions, a 0-1 law holds for directional transience. That is, the walk is either almost surely transient to the right, almost surely transient to the left, or almost surely recurrent. In the recurrent case, $v = 0$. We provide two abstract characterizations of ballisticity under the following additional assumption.

(C4) For P–a.e. environment ω, $\lim_{n\to\infty} X_n = \infty$, P_ω^0–a.s.

By symmetry, our characterizations also handle the case where the walk is transient to the left, and thus by the 0-1 law of [16], completely characterize the regime $v \neq 0$ for all measures P satisfying (C1), (C2), and (C3).

The first characterization strengthens one given by Brémont, who showed (see [33, Theorem 3.7], [18, Proposition 9.1]) that for a walk that is transient to the right, $v > 0$ if and only if the annealed expected time to reach $[1, \infty)$ is finite. Brémont's works used an ellipticity assumption that is too strong to apply to our Dirichlet environments. We therefore prove the lemma without the assumption.

Lemma 2.3.1. *Let P be a probability measure on $\Omega_{\mathbb{Z}}$ satisfying (C1), (C2), (C3), and (C4). Then $v > 0$ if and only if $\mathbb{E}^0[T_{\geq 1}] < \infty$, where $T_{\geq 1}$ is the first time the walk hits $[1, \infty)$.*

Jump to proof.

This characterization is quite natural, given that in the nearest-neighbor case we in fact have the identity $v = 1/\mathbb{E}^0[T_{\geq 1}]$, where the fraction is understood to be 0 if the denominator is infinite. However, although it is natural, we do not know a way to check it directly in the (L, R) case, even for Dirichlet environments. We therefore present a new abstract criterion for ballisticity, showing that the walk is ballistic if and only if the annealed expected number of returns to the origin is finite.

Lemma 2.3.2. *Let P be a probability measure on $\Omega_{\mathbb{Z}}$ satisfying (C1), (C2), (C3), and (C4). Then $v > 0$ if and only if $\mathbb{E}^0[N_0] = E[E_\omega^0[N_0]] < \infty$.*

Jump to proof.

To do this, we define a "walk from $-\infty$ to ∞" in a typical environment where transience to the right holds. By an argument using Birkhoff's Ergodic theorem, the almost-sure limiting speed of this bi-infinite walk is the reciprocal of the expected amount of time it spends at 0 under an appropriate annealed measure. Thus, the limiting speed is zero precisely when this expectation is infinite. We show that this expectation in turn is infinite if and only if the expected amount of time at 0 for a walk started at 0 is infinite.

Thus, the question of ballisticity is reduced to the integrability of the "Green function" $E^0_\omega[N_0]$ under the measure on environments. We devote Section 4.2 to answering this question in the case of our Dirichlet measure $P_{\mathcal{G}}$. In fact, we go further and characterize integribility of $E^0_\omega[N_0]^s$ for any $s > 0$ in terms of the Dirichlet parameters, and in particular in terms of κ_0 and κ_1.

2.3.2 Finite traps of RWDE on $\mathbb{Z}$

In nearest-neighbor RWDE, the underlying directed graph has an edge from x to y precisely when x and y are adjacent. There, it can easily be shown that the worst finite traps are just pairs of vertices, and so κ_0 has an explicit formula as a minimum of d different sums of edge weights. By contrast, our model encompasses many underlying directed graphs (even before assignment of weights). For each underlying directed graph there is a different formula for κ_0 as a minimum of finitely many sums (as we show in Proposition 4.2.2), but we do not have a general method to find the formula given a particular underlying directed graph. This is because we have no simple general way to know what the worst finite traps look like. However, we find the formula in several examples in Appendix B, and show in Proposition 4.2.1 that κ_0 can be calculated directly from L, R, and the specific values of the α_i, even without a general formula in terms of the α_i.

This parameter κ_0 plays an important role in the integrability of $E^0_\omega[N_0]$. For a set $S \subseteq \mathbb{Z}$, and for $x \in S$, define N^S_x to be the amount of time a walk spends at x before leaving S for the first time (we always have $N^S_x \leq N_x$). We define κ_0 as an infimum of sums of edge weights. This infimum is over an infinite set, but once we can show that it is actually a minimum, the following theorem follows almost immediately from [29, Theorem 1].

Theorem 2.3.3. *For $s > 0$, the following are equivalent:*

(a) $\kappa_0 \leq s$.

(b) For all sufficiently large M, $E_{\mathcal{G}}\left[E_\omega^0\left[N_0^{[-M,0]}\right]^s\right] = \infty$.

(c) For some $M \geq 0$, $E_{\mathcal{G}}\left[E_\omega^0\left[N_0^{[-M,M]}\right]^s\right] = \infty$.

Jump to proof.

Letting $s = 1$, the implication (a)⇒(b) or (a)⇒(c) shows that if $\kappa_0 \leq 1$, then $\mathbb{E}_{\mathcal{G}}^0[N_0] = \infty$, which by Lemma 2.3.2 implies $v = 0$. We include condition (b) because the implication (a)⇒(b) allows one to arrive at the same conclusion using Lemma 2.3.1 (by showing that $\kappa_0 \leq 1$ implies $\mathbb{E}_{\mathcal{G}}^0[T_{\geq 1}] = \infty$).

2.3.3 Large-scale backtracking of RWDE on $\mathbb{Z}$

While κ_0 controls the moments of the quenched expected amount of time the walk spends at 0 before exiting a finite region of the graph, the parameter $\kappa_1 = d^+ - d^-$ controls, in the same way, the moments of the quenched expected number of times the walk traverses arbitrarily large regions of the graph. For $x < y \in \mathbb{Z}$, we define the following functions of a walk $\mathbf{X}$:

- $N_{x,y}(\mathbf{X}) = \#\Big\{n \in \mathbb{N}_0 : X_n = x, \sup\{j < n : X_j = y\} > \sup\{j < n : X_j = x\}\Big\}$ is the number of times the walk hits x after more recently having hit y, or the number of "trips from y to x".

- $N'_{x,y}(\mathbf{X}) := \#\Big\{n \in \mathbb{N}_0 : X_n \leq x, \sup\{j < n : X_j \geq y\} > \sup\{j < n : X_j \leq x\}\Big\}$ is the number of trips leftward across $[x, y]$.

Again, we write these as $N_{x,y}$ and $N'_{x,y}$ if we can do so without ambiguity. Note that $N_x \geq N_{x,y}$, and also $N'_{x,y} \geq N_{x,y}$. We prove the following theorem.

Theorem 2.3.4. *Let $\kappa_1 > 0$, so that the walk is transient to the right. Then, if $s > 0$, the following are equivalent:*

(a) $\kappa_1 > s$.

(b) There is an $M \geq 0$ such that for all $x, y \in \mathbb{Z}$ with $y - x \geq M$, $E_{\mathcal{G}}\left[E_\omega^0[N'_{x,y}]^s\right] < \infty$.

(c) There exist $x < y \in \mathbb{Z}$ such that $E_{\mathcal{G}}\left[E_\omega^0[N_{x,y}]^s\right] < \infty$.

Jump to proof.

The proof of Theorem 2.3.4 is long and naturally divides into two parts, so we prove the parts separately as Proposition 4.2.4 and Proposition 4.2.5. Letting $s = 1$, the contrapositive of the implication (c)$\Rightarrow$(a) tells us that if $\kappa_1 \le 1$, then $\mathbb{E}_{\mathcal{G}}^0[N_0] = \infty$, which by Lemma 2.3.2 implies $v = 0$.

We informally describe some of the ideas behind the proof of Theorem 2.3.4 here, assuming for the sake of simplicity that we are interested in the case $s = 1$. To understand the relevance of κ_1 in determining ballisticity, we can look at the graph $\mathcal{G}_+$, a modified version of the graph $\mathcal{G}$, which is half-infinite and has zero divergence. We define the graph formally in Section 4.2.2, but we show it now in Figure 2.3. A lemma from [22] states that under $P_{\mathcal{G}_+}$, the quantity $P_\omega^0(\tilde{T}_0 = \infty)$ is distributed as a beta random variable with first parameter κ, and thus its reciprocal has finite moments up to (but not including) κ_1.

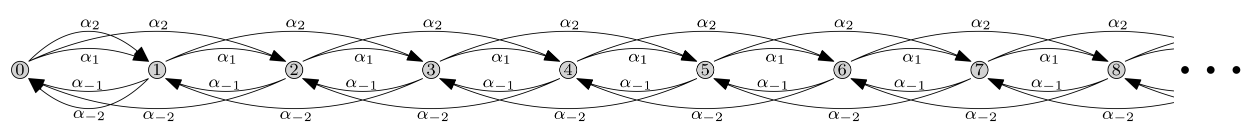

Figure 2.3. Graph $\mathcal{G}_+$.

If $\kappa_1 \le 1$, then in the modified graph, one expects an infinite number of returns to 0. We use this fact to prove that in the original graph $\mathcal{G}$, one expects an infinite number of returns from $[1, R]$ to $(-\infty, 0]$. To do this, we couple transition probabilities in the original graph with corresponding transition probabilities in the modified graph. See Figure 2.4.

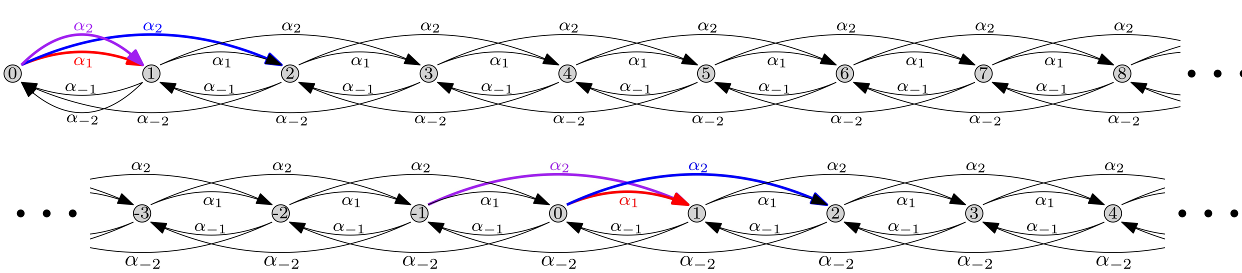

Figure 2.4. A coupling between transition probabilities in $\mathcal{G}$ and $\mathcal{G}_+$.

The coupling is tricky, since the transition probabilities in the two graphs have different dependence relationships, and are also not distributed in the same way. However, the transition probabilities are all mixtures of independent beta random variables. By looking at the first parameters of these beta random variables, we are thus able to define "coupling event" of positive probability, on which the beta random variables used for $\mathcal{G}$ are all smaller than the corresponding ones used for $\mathcal{G}_+$. This coupling event is independent of the transition probabilities in the graph $\mathcal{G}_+$. Due to the differences in dependence relationships, the coupling event does not automatically give us transition probabilities in $\mathcal{G}$ that are all smaller than the corresponding ones in $\mathcal{G}_+$. However, a certain random ordering of head vertices in the coupling (according to probability of not backtracking to 0) allows us to get around the dependence problems. We are ultimately able to show that on the coupling event, the maximum probability in the original graph $\mathcal{G}$, starting at 0 or any site to its left, of stepping to the right and never returning, is bounded above by a multiple of the corresponding probability in the modified graph $\mathcal{G}_+$. Since the coupling event has positive probability and is independent of probabilities in $\mathcal{G}_+$, the fact that one expects an infinite number of returns to 0 in $\mathcal{G}_+$ then implies that one also expects an infinite number of returns to $[1-L, 0]$ in $\mathcal{G}$. A somewhat more careful analysis gives an infinite expected number of returns to 0 as well as an infinite expected number of traversals of $[-M, 0]$ for any $M > 0$. Since this expectation is infinite regardless of where the walk starts, translation invariance gives us the theorem for the case $\kappa_1 \leq 1$.

The case where $\kappa_1 > 1$ turns out not to be necessary for our characterization of ballisticity. However, it gives us a better understanding of the way in which the parameter κ_1 affects large-scale backtracking. To handle that case, we again use a coupling-type argument. The vertex in $[1, R]$ with the best quenched probability of never backtracking to 0 has the same distribution in $\mathcal{G}$ and $\mathcal{G}_+$, as does the non-backtracking probability at this vertex. Thanks to the lemma from [22] about $\mathcal{G}_+$, we are able to get good control of moments of the reciprocal of this probability. We then choose M large enough that a walk started to the right of M is highly likely to hit a vertex with a non-backtracking probability as good as that of the best vertex in $[1, R]$ before backtracking to 0. An application of Hölder's inequality gives us the result we need.

2.3.4 Full ballisticity characterization for RWDE on $\mathbb{Z}$

Combining the theorems stated so far, we can see that if $\kappa_0 \leq 1$, then the walk is not ballistic due to finite trapping, and that if $\kappa_1 \leq 1$, then the walk is not ballistic due to large-scale backtracking. We would like to show that if both parameters are greater than 1, then the walk is ballistic. For every environment ω on $\mathbb{Z}$ and every $M > 0$, we have

$$E_\omega^0[N_0] = E_\omega^0\left[N_0^{[-M,M]}\right] E_\omega^0\left[\#\begin{Bmatrix}\text{Times exiting } [-M,M] \\ \text{and then returning to } 0\end{Bmatrix}\right]. \tag{2.1}$$

The first expectation on the right relates to finite trapping, and the second to large-scale backtracking. By Theorem 2.3.3, the term $E_\omega^0\left[N_0^{[-M,M]}\right]$ has finite moments up to κ_0 under $P_{\mathcal{G}}$ for M sufficiently large. And the number of times exiting $[-M, M]$ and returning to 0 is between $N_{-M-1,0}+N_{0,M+1}$ and $N'_{-M-1,0}+N'_{0,M+1}$, so the term $E_\omega^0\left[\#\begin{Bmatrix}\text{Times exiting } [-M,M] \\ \text{and then returning to } 0\end{Bmatrix}\right]$ has finite moments up to κ_1 by Theorem 2.3.4. If the two terms on the right side of (2.1) were independent under $P_{\mathcal{G}}$, we could conclude that $E_{\mathcal{G}}[E_\omega^0[N_0]^s] < \infty$ if and only if $s < \min(\kappa_0, \kappa_1)$. However, they are not independent. We therefore ask whether it is possible that the phenomena of finite trapping and large-scale backtracking may conspire together to prevent ballisticity, even if neither is strong enough to do it on its own. This question is of interest for general RWRE on $\mathbb{Z}$ with bounded jumps.

Question 2.3.1. *Let P be a probability measure on $\Omega_{\mathbb{Z}}$ satisfying (C1), (C2), (C3), and (C4), under which both terms on the right of (2.1) have finite expectation for all M; that is, $E\left[E_\omega^0\left[N_0^{[-M,M]}\right]\right] < \infty$ and $E\left[E_\omega^0\left[\#\begin{Bmatrix}\text{Times exiting } [-M,M] \\ \text{and then returning to } 0\end{Bmatrix}\right]\right] < \infty$. Does it necessarily follow that $\mathbb{E}^0[N_0] = E[E_\omega^0[N_0]] < \infty$ (and thus that the walk is ballistic)?*

We are able to answer this question in the affirmative for our Dirichlet model. In fact, we characterize the finiteness of all moments of $E_\omega^0[N_0]$ under $P_{\mathcal{G}}$.

Theorem 2.3.5. *Assume $\kappa_1 > 0$. Then $E_{\mathcal{G}}\left[(E_\omega^0[N_0])^s\right] < \infty$ if and only if $s < \min(\kappa_0, \kappa_1)$.*

Jump to proof.

Using a symmetry argument for the case $\kappa_1 < 0$ and combining this result with Lemma 2.3.2, we get a complete characterization of ballisticity.

Theorem 2.3.6. *The walk is ballistic if and only if* $\min(\kappa_0, |\kappa_1|) > 1$.

Jump to proof.

We describe our proof Theorem 2.3.5 informally, focusing on the case where $s = 1$. We assume transience to the right. Due to Theorems 2.3.3 and 2.3.4, we need only address the case $\min(\kappa_0, \kappa_1) > 1$. As in the proof of Theorem 2.3.4 when $\kappa_1 > 1$, we focus on the "best site" in a set of R consecutive sites—the site in the set that has the highest quenched probability of not backtracking to the left of that set. Here, we compare such a "best site" in $[1, R]$ in $\mathcal{G}_+$ with "best sites" in strips of length R that lie just to the right of strips of length m in $\mathcal{G}$, where m is an integer large enough that all vertices in $[1, m]$ communicate within $[1, m]$. We take M to be a large multiple of m, containing many strips of length m. Then the "best strip" is the one in which all sites can most easily reach all sites just to the right of the strip. We depict the comparison in Figure 2.5

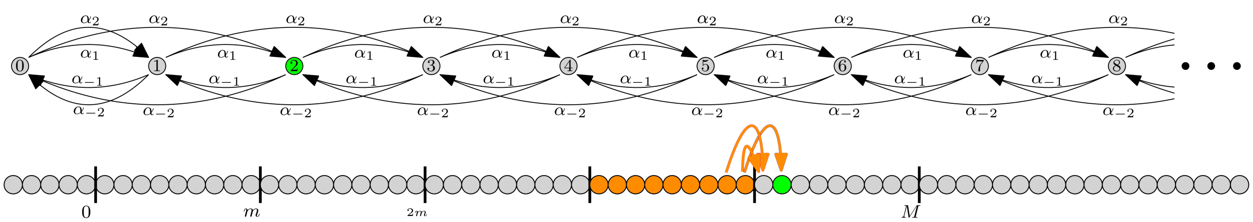

Figure 2.5. The comparison between best sites in $\mathcal{G}_+$ and $\mathcal{G}$.

We now give a simplified, informal version of the comparisons used in the argument. We will give the argument in full detail later on. We note that for any environment ω, the number of visits to 0 is a geometric random variable under P^0_ω. Thus,

$$E^0_\omega[N_0] = \frac{1}{P^0_\omega(\tilde{T}_0 = \infty)}.$$

From this, we get

$$E_\omega^0[N_0] \;\le\; \frac{1}{P_\omega^0\begin{pmatrix}\text{Reach best strip}\\ \text{on 1st excursion}\end{pmatrix}} \cdot \frac{1}{P_\omega^{\text{b. strip}}\begin{pmatrix}\text{Exit strip}\\ \text{at best site}\end{pmatrix}} \cdot \frac{1}{P_\omega^{\text{b. site}}(\text{Never backtrack})}$$

$$= \sum_{\text{strips}} \frac{\mathbb{1}_{\text{strip is best}}}{P_\omega^0\begin{pmatrix}\text{Reach strip on}\\ \text{1st excursion}\end{pmatrix} \cdot P_\omega^{\text{b. strip}}\begin{pmatrix}\text{Exit strip}\\ \text{at best site}\end{pmatrix} \cdot P_\omega^{\text{b. site}}(\text{Never backtrack})}$$

$$\le \frac{1}{P_\omega^{\text{b. strip}}\begin{pmatrix}\text{Exit strip}\\ \text{at best site}\end{pmatrix}} \sum_{\text{strips}} \frac{1}{P_\omega^0\begin{pmatrix}\text{Reach strip on}\\ \text{1st excursion}\end{pmatrix} \cdot P_\omega^{\text{b. site}}(\text{Never backtrack})}.$$

We want to apply Hölder's inequality to show that the left hand side has finite expectation under P. We can show that $P_\omega^0(\text{Reach strip on 1st excursion})^{-1}$ has finite moments up to $\min(\kappa_0, d_+) \ge \min(\kappa_0, \kappa_1)$. And $P_\omega^{\text{b. site}}(\text{Never backtrack})^{-1}$ has finite moments up to κ_1 by comparison with the modified graph. The two terms in the denominator of the sum are independent and the sum has finitely many terms, so the sum has finite moments up to $\min(\kappa_0, \kappa_1) > 1$. By choosing M large enough that there are many strips of size m, and the fraction in front of the sum has sufficiently high finite moments, we are able to apply Hölder's inequality and get the result.

2.4 Acceleration

The purpose of Chapter 5 is to set up a mechanism for comparing our results on ballisticity with those known for RWDE in higher dimensions. The concept of finite trapping—that is, the existence of finite regions in which a walk is expected to spend an infinite amount of time before exiting for the first time—is not tied to any particular dimension. Large-scale backtracking, on the other hand, is a very one-dimensional concept. Nevertheless, the idea that, *a priori*, there could be global factors that cause a directionally transient walk to have zero speed regardless of finite trapping effects seems intuitively intelligible in all dimensions, and it would be desirable to formulate in a precise way the question of whether such large-scale slowing exists.

We give one such formulation here, and prove an answer to the question for our model in Section 6. Our formulation is inspired by a remark by Sabot and Tournier in [15] regarding one of the implications of the method Bouchet used in [14]. In the process of characterizing directional transience for the case $d \geq 3$, Bouchet showed that in the zero-speed case $\kappa_0 \leq 1$, accelerating the walk through finite traps yields a continuous-time walk that is ballistic. To do this, she considered, for each environment ω, a continuous-time random walk on $\mathbb{Z}^d$ with exponential waiting times, whose jump rate from x to y is a multiple of $\omega(x, y)$ that is fixed for each x and depends only on the environment within a fixed radius of x. Because the jump rates are proportional to $\omega(x, y)$, the next vertex visited is distributed according to the measure $\omega(x, \cdot)$. Thus, Bouchet's continuous-time walk started at x follows the path of a walk distributed according to P_ω^x. Accelerating the jump rates in proportion to the strength of finite traps in the quenched environment allows the continuous-time walk to avoid spending a large amount of time in traps of bounded size while preserving larger-scale behaviors. Bouchet showed that for nearest-neighbor RWDE in $\mathbb{Z}^d$, $d \geq 3$, an appropriate acceleration scheme could always yield a ballistic walk.

Bouchet's motivation for introducing this acceleration scheme was not to study ballisticity, but to study transience. The acceleration scheme yields a walk with for which there is an invariant measure for the environment from the point of view of the particle that is absolutely continuous with respect to the distribution of the initial environment. This yields a 0-1 law for directional transience, which necessarily also applies to the original (unaccelerated) walk. As a byproduct, however, it implies ballisticity of the accelerated walk, and as Sabot and Tournier remark [15, Remark 7], Bouchet's results demonstrate that finite trapping is in some sense the only obstacle to ballisticity, because accelerating through them causes the walk to be ballistic.

Motivated by Bouchet's work and by this remark from [15], we define the term "essential slowing" to describe a situation where no acceleration scheme like Bouchet's can yield a ballistic walk. We show that in our model, essential slowing is equivalent to large-scale backtracking being significant enough to cause zero speed. Thus, unlike in the higher-dimensional case Bouchet studied, essential slowing can occur in our model, and in fact can occur with or without finite traps.

In order to precisely define essential slowing, we must define a specific class of continuous-time random walks. Say a *Bouchet acceleration function* on $\mathbb{Z}^d$ is a measurable function $\mathcal{A}$ from the space $\Omega_{\mathbb{Z}^d}$ of environments on $\mathbb{Z}^d$ to the space of distributions of positive random variables, where $\mathcal{A}(\omega)$ only depends on $\omega^{[-M,M]}$ for some positive integer M, a condition which ensures that the acceleration is only based on traps of bounded size.

For any environment ω on $\mathbb{Z}^d$, and any point $x \in \mathbb{Z}^d$, let $\theta^x\omega$ be the shifted environment defined by $\theta^x\omega(a,b) = \omega(x+a, x+b)$. Now for an environment ω on $\mathbb{Z}^d$, a point $x \in \mathbb{Z}^d$, and a Bouchet acceleration function $\mathcal{A}$, we can define a continuous-time Markov chain $(X_t)_{t\geq 0}$ on $\mathbb{Z}^d$ where $X_0 = x$ almost surely, and the walk stays at x for a time distributed according to $\mathcal{A}(\theta^x\omega)$ before jumping to a point chosen according to the transition probabilities given by ω. Whenever the process hits a point $a \in \mathbb{Z}$, it remains there for an amount of time distributed according to $\mathcal{A}(\theta^a\omega)$ and independent of all other information about the history of the process before jumping to a point chosen according to ω. Let $P^x_{\omega,\mathcal{A}}$ be the law of this process. Thus, the sequence of vertices visited (up to immediate repetitions of the same vertex, if there are self-loops) has the same distribution under P^x_ω and $P^x_{\omega,\mathcal{A}}$. We let $\mathbb{P}^x_{\mathcal{G},\mathcal{A}}$ be the corresponding annealed law. As in the discrete-time case, there is necessarily a $\mathbb{P}^x_{\mathcal{G},\mathcal{A}}$-almost-sure limiting velocity $v(\mathcal{A}) \geq 0$, at least when the walk is directionally transient.

Remark 2.4.1. If $\mathcal{A} \equiv \delta_1$ (that is, if $\mathcal{A}(\omega)$ is the distribution of a degenerate random variable that is deterministically 1), then the continuous-time walk $(X_t)_{t\geq 0}$, distributed according to $P^x_{\omega,\mathcal{A}}$ and observed only at integer times $t = 0, 1, 2\ldots$, follows precisely the same law as the walk $(X_n)_{n=0}^\infty$ under P^x_ω. Thus, our discrete model may be thought of as a special case of this continuous-time model.

In a sense, the most natural acceleration scheme is one where $\mathcal{A}(\omega)$ is the distribution of an exponential random variable whose expectation depends on ω, which allows the accelerated process to be a continuous time Markov chain. Indeed, these are the functions Bouchet considered. However, our definition allows for more general distributions in order to incorporate the possibility $\mathcal{A} \equiv \delta_1$, which allows proofs to more conveniently cover accelerated and unaccelerated walks at the same time.

An important feature of Bouchet acceleration functions is that because they only depend on the environment a finite distance from the origin, slowing of the walk can be corrected by an appropriate Bouchet acceleration function only when it is due to finite traps. Thus, these functions can test whether factors other than finite traps are by themselves sufficient to prevent ballisticity.

Definition 2.4.2. A probability measure P on $\Omega_{\mathbb{Z}^d}$ under which the walk is almost surely transient in some direction is said to have *essential slowing* if, for any Bouchet acceleration function $\mathcal{A}$, it is the case that $v(\mathcal{A}) = 0$.

In contrast to the nearest-neighbor case $d \geq 3$, nearest-neighbor RWDE on $\mathbb{Z}$ do allow for essential slowing. In fact, we show that in our bounded-jump model, essential slowing occurs exactly when $\kappa_1 \leq 1$. That is, essential slowing corresponds to large-scale backtracking.

Theorem 2.4.1. *Assume $\kappa_1 \neq 0$. Then $P_{\mathcal{G}}$ has essential slowing if and only if $|\kappa_1| \leq 1$.*

Jump to proof.

The proof is a fairly straightforward translation of our previous results and arguments to the setting of accelerated walks.

We may also contrast our model with the one-dimensional nearest-neighbor model: suppose the transition probabilities $\omega(x, x+1)$ are all beta random variables with parameters (α_1, α_{-1}), and that $\alpha_1 \neq \alpha_{-1}$, so that the walk is transient. Then finite traps occur if and only if $\alpha_1 + \alpha_{-1} \leq 1$, but by Theorem 2.4.1, essential slowing occurs if and only if $|\alpha_1 - \alpha_{-1}| \leq 1$. Hence finite traps imply essential slowing. Unlike the nearest-neighbor models of $\mathbb{Z}^d$, both for $d = 1$ and for $d \geq 3$, our model allows essential slowing and finite traps to each occur with or without the other. We illustrate this difference in Tables 2.1, 2.2, and 2.3.

A well known conjecture states that for iid, uniformly elliptic RWRE in $d \geq 2$, directional transience implies ballisticity. Sabot and Tournier [15, Remark 7] suggest a related conjecture for all i.i.d. RWRE in $d \geq 2$ (whether uniformly elliptic or not), which our definition of essential slowing allows us to make explicit.

Conjecture 2.4.2. *For irreducible, iid, directionally transient RWRE in $d \geq 2$, essential slowing is impossible. That is, there always exists a Bouchet acceleration function $\mathcal{A}$ such that $v(\mathcal{A}) \neq 0$.*

Table 2.1. Nearest-neighbor RWDE on $\mathbb{Z}$

	Essential Slowing	No Essential Slowing
Finite Traps	✓	✗
No Finite Traps	✓	✓

Table 2.2. Nearest-neighbor RWDE on $\mathbb{Z}^d$, $d \geq 3$

	Essential Slowing	No Essential Slowing
Finite Traps	✗	✓
No Finite Traps	✗	✓

Table 2.3. RWDE on $\mathbb{Z}$ with bounded jumps

	Essential Slowing	No Essential Slowing
Finite Traps	✓	✓
No Finite Traps	✓	✓

Because uniformly elliptic environments have no finite trapping effects at all in the sense that the quenched expected exit time from any box of bounded size is almost surely bounded, there is reason to suspect that the above conjecture implies the classic conjecture about uniformly elliptic RWRE in $d \geq 2$. However, for regimes that are not uniformly elliptic but have finite annealed expected exit times from every box, one could still ask a question analogous to Question 2.3.1. If every box has finite annealed expected exit time and there is no essential slowing, can one conclude that the walk is ballistic?

Question 2.4.1. *Let P be a probability measure on $\Omega_{\mathbb{Z}^d}$ satisfying (C1), (C2), and (C3), and suppose there is an almost-sure limiting direction. Suppose essential slowing does not occur, and also that* $E\left[E_\omega^0\left[\#\left\{\begin{matrix}\textit{Times exiting } [-M,M]^d \\ \textit{and then returning to 0}\end{matrix}\right\}\right]\right] < \infty$ *for all M. Does it necessarily follow that the walk is ballistic?*

In one dimension, we suspect that this question is equivalent to Question 2.3.1, and our results show that it is equivalent for RWDE.

3. DIRECTIONAL TRANSIENCE AND RECURRENCE

In this chapter, we prove our results about directional transience, first a 0-1 law for RWRE on $\mathbb{Z}^2$ with bounded jumps, and then our characterization of directional transience in a given direction for RWDE on $\mathbb{Z}^d$ with bounded jumps.

3.1 0-1 Law

First, we extend the 0-1 law of [23] and [31] for RWRE on $\mathbb{Z}^2$ to the case where bounded jumps are allowed.

Theorem (Theorem 2.1.1)**.** *Let $d = 2$, and let assumptions (C1), (C2), and (C3) hold, and let $\ell \in S^1$. Then $\mathbb{P}^0(A_\ell) \in \{0, 1\}$, where A_ℓ is the event $\lim_{n\to\infty} X_n \cdot \ell = \infty$.*

Proof of Theorem 2.1.1. For $y \in \mathbb{Z}^2$ and a path $\gamma = (x_0, x_1, \ldots, x_n)$ (which is a path of length n, and is a loop if $x_n = x_0$), define $y + \gamma := (y + x_0, y + x_1, \ldots, y + x_n)$. This is simply a space shift of the path γ. For a path $\gamma = (x_0, x_1, \ldots, x_n)$, we will talk about the *annealed probability of* γ, or the *probability that* $\mathbf{X}$ *takes* γ. This simply means

$$\mathbb{P}^{x_0}(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n).$$

Say that γ is a *possible path* if it has positive annealed probability. Note that a loop (x_0) of zero length has annealed probability 1, since $\mathbb{P}^{x_0}(X_0 = x_0) = 1$.

Now by assumption (C2), there is a path of positive annealed probability connecting any two points. Let M be large enough that for any vertex x in a closed unit disc of radius $2R$ centered at 0, there is a path of positive probability from 0 to x with length no more than M.

By Kalikow's 0-1 law (see Appendix A), $\mathbb{P}^0(A_\ell \cup A_{-\ell}) \in \{0, 1\}$. Thus, it suffices to show that $\mathbb{P}^0(A_\ell)\mathbb{P}^0(A_{-\ell}) = 0$ under the assumption that $\mathbb{P}^0(A_\ell \cup A_{-\ell}) = 1$. From the proof of Kalikow's 0-1 law, one gets that $\mathbb{P}^0(A_\ell) > 0$ if and only if $\mathbb{P}^0(T_{<0} = \infty) > 0$. Thus, it suffices to show

$$\mathbb{P}^0(T_{<0} = \infty)\mathbb{P}^0(T_{>0} = \infty) = 0. \tag{3.1}$$

For $a, b \in \mathbb{R}$, define the event

$$G_a^b := \begin{cases} \{T_{\geq b} < T_{<a}\} & \text{if } b > a; \\ \{T_{\leq b} < T_{>a}\} & \text{if } b < a \end{cases}.$$

Note that for fixed a,

$$\lim_{b \to \infty} G_a^b = \{T_{<a} = \infty\}; \quad \lim_{b \to -\infty} G_a^b = \{T_{>a} = \infty\}. \tag{3.2}$$

Let $L > 0$, and fix a unit vector $\ell^\perp$ perpendicular to ℓ. Choose a sequence $z_L \in \mathbb{Z}^2$ such that

- $x_L := z_L \cdot \ell \geq 2L$,
- With positive $\mathbb{P}^0$-probability, $X_{T_{\geq 2L}} = z_L$, and
- $z_L \cdot \ell^\perp$ is a median of the distribution of $X_{T_{\geq 2L}} \cdot \ell^\perp$ under the measure $\mathbb{P}^0(\cdot | G_0^{2L})$. That is, $\mathbb{P}^0(X_{T_{\geq 2L}} \cdot \ell^\perp > z_L \cdot \ell^\perp | G_0^{2L}) \leq \frac{1}{2}$ and $\mathbb{P}^0(X_{T_{\geq 2L}} \cdot \ell^\perp < z_L \cdot \ell^\perp | G_0^{2L}) \leq \frac{1}{2}$.

Due to the allowance of jumps, z_L may not be uniquely defined for each L—for example, if $\ell = (1, 0)$ and a jump of two steps to the right is possible, then $(2L, h)$ and $(2L+1, h)$ would both be candidates for z_L for some h—but one may, for instance, always take the candidate with the smallest ℓ component. Now consider two independent random walks $\mathbf{X}^1 = (X_n^1)_n$ and $\mathbf{X}^2 = (X_n^2)_n$ moving in the same environment, with the first walk starting at 0 and the second starting at z_L. For $\omega \in \Omega_{\mathbb{Z}^2}$ and $a, b \in \mathbb{Z}^2$, let $P_\omega^{a,b}$ be the product measure $P_\omega^a \times P_\omega^b$ on the set $(\mathbb{Z}^2)^{\mathbb{N}_0} \times (\mathbb{Z}^2)^{\mathbb{N}_0}$ with typical element $(\mathbf{X}^1, \mathbf{X}^2)$. Let $\mathbb{P}^{a,b}$ be the corresponding annealed measure.

We consider the "strip traversal event" $G_0^{2L} \times G_{x_L}^0$, which is roughly the event that both walks cross the strip $\{0 \leq x \cdot \ell \leq 2L\}$ before leaving it; the walk starting at 0 is in G_0^{2L}, while the walk starting at z_L is in $G_{x_L}^0$. Zerner shows[1] in [31] that

$$\mathbb{P}^0(T_{<0} = \infty)\mathbb{P}^0(T_{>0} = \infty) = \lim_{L\to\infty} \mathbb{P}^{0,z_L}(G_0^{2L} \times G_{x_L}^0). \tag{3.3}$$

Now consider the following three subsets of the strip traversal event:

- $\mathcal{O}_L$, the *opposite-sides event*. This is the event that $\mathbf{X}^1 \in G_0^{2L}$, $\mathbf{X}^2 \in G_{x_L}^{-L}$, and $\left[(X^1_{T_{\geq 2L}} - z_L) \cdot \ell^\perp\right] \left[X^2_{T_{\leq 0}} \cdot \ell^\perp\right] < 0$.
- $\mathcal{I}_L$, the *intersection event*. This is the event that $\mathbf{X}^1 \in G_0^{2L}$, $\mathbf{X}^2 \in G_{x_L}^{-L}$, and for some $0 \leq m \leq T_{\geq 2L}(\mathbf{X}^1)$, $0 \leq n \leq T_{\leq 0}(\mathbf{X}^2)$, $X^1_m = X^2_n$.
- $\mathcal{P}_L$, the *proximity event*. This is the event that $\mathbf{X}^1 \in G_0^{2L}$, $\mathbf{X}^2 \in G_{x_L}^{-L}$, and for some $0 \leq m \leq T_{\geq 2L}(\mathbf{X}^1)$, $0 \leq n \leq T_{\leq 0}(\mathbf{X}^2)$, $|X^1_m - X^2_n| \leq 2R$.

Clearly $\mathcal{I}_L \subset \mathcal{P}_L$. We claim

$$G_0^{2L} \times G_{x_L}^{-L} = \mathcal{O}_L \cup \mathcal{P}_L = \mathcal{O}_L \cup \mathcal{P}_L \cup \mathcal{I}_L. \tag{3.4}$$

The events $\mathcal{O}_L$ and $\mathcal{P}_L$ are each specified to be contained in the event $G_0^{2L} \times G_{x_L}^0$, so their union is as well. Now assume $\mathbf{X}^1 \in G_0^{2L}$ and $\mathbf{X}^2 \in G_{x_L}^0$. We will show that either $\mathcal{O}_L$ or $\mathcal{P}_L$ occurs. Let $\overline{\pi}^1$ be the continuous linear interpolation of the path taken by $\mathbf{X}^1$, and let $\overline{\pi}^2$ be the continuous linear interpolation of the path taken by $\mathbf{X}^2$. Let α_2 be the last point in $\mathbb{R}^2$ where $\overline{\pi}^2$ crosses the line $\{x \cdot \ell = 2L\}$. Let β_1 be the first point where $\overline{\pi}^1$ crosses $\{x \cdot \ell = 2L\}$, and let β_2 be the first point where $\overline{\pi}^2$ crosses $\{x \cdot \ell = 0\}$. Let z'_L be the point on the line $\{x \cdot \ell = 2L\}$ with $(z_L - z'_L) \cdot \ell^\perp = 0$ (thus, $z_L = z'_L + (x_L - 2L)\ell$). Note α_2, β_1, β_2, and z'_L need not be in $\mathbb{Z}^d$.

[1]↑Zerner actually shows $\mathbb{P}^0(T_{<0} = \infty)\mathbb{P}^0(T_{>0} = \infty) \leq \liminf_{L\to\infty} \mathbb{P}^{0,z_L}(G_0^{2L} \times G_{x_L}^0)$, and this is all that is needed for his argument and ours. However, a brief and straightforward addition to Zerner's argument would show that the liminf is actually a limit and that equality holds, so we write it that way for cosmetic reasons.

To show that either $\mathcal{O}_L$ or $\mathcal{P}_L$ occurs, we will assume $\mathcal{O}_L$ does not occur and prove that $\mathcal{P}_L$ must occur. If $\mathcal{O}_L$ does not occur, then $(X^1_{T_{\geq 2L}} - z_L) \cdot \ell^\perp$ and $X^2_{T_{\leq 0}} \cdot \ell^\perp$ are either both positive or both negative, or else at least one is 0. If $(X^1_{T_{\geq 2L}} - z_L) \cdot \ell^\perp = 0$, then $\mathcal{P}_L$ occurs, because $X_{T_{\geq 2L}}$ and z_L have the same $\ell^\perp$ component and both have ℓ component between $2L$ and $2L + R$. Similarly, if $X^2_{T_{\leq 0}} \cdot \ell^\perp = 0$, then $\mathcal{P}_L$ occurs.

Now suppose $(X^1_{T_{\geq 2L}} - z_L) \cdot \ell^\perp$ and $X^2_{T_{\leq 0}} \cdot \ell^\perp$ are both nonzero and have the same sign. Without loss of generality, we may assume both are positive (otherwise, rename the directions $\ell^\perp$ and $-\ell^\perp$). To show that $\mathcal{P}_L$ occurs, we must show that for some $0 \leq m \leq T_{\geq 2L}(\mathbf{X}^1)$ and for some $0 \leq n < T^2_0$, $|X^1_m - X^2_n| \leq 2R$.

First, suppose that $\beta_2 \cdot \ell^\perp < 0$; this situation is depicted in Figure 3.1. Then $X^2_{T_{\leq 0}-1} \cdot \ell > 0$ and $X^2_{T_{\leq 0}-1} \cdot \ell^\perp < 0$, but $X^2_{T_{\leq 0}} \cdot \ell < 0$ and $X^2_{T_{\leq 0}} \cdot \ell^\perp > 0$. In one step, the walker that started at z_L crosses the line $\{x \cdot \ell = 0\}$ and the line $\{x \cdot \ell^\perp = 0\}$. It follows that $X^2_{T_{\leq 0}-1}$ must be within a radius R of 0, and the event $\mathcal{P}_L$ occurs. Similarly, if $(\beta_1 - z_L) \cdot \ell^\perp < 0$, then $X^1_{T_{\geq 2L}-1}$

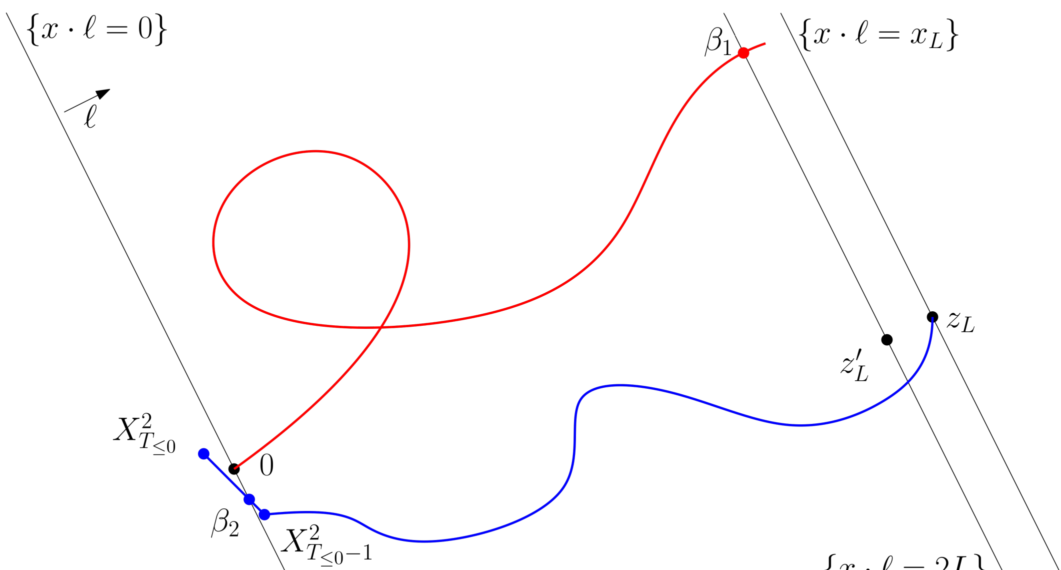

Figure 3.1. $(X^1_{T_{\geq 2L}} - z_L) \cdot \ell^\perp > 0$ and $X^2_{T_{\leq 0}} \cdot \ell^\perp > 0$, but $\beta_2 \cdot \ell^\perp < 0$.

must be within a radius R of z'_L. Since z'_L is within distance R from z_L, we conclude that $X^1_{T_{\geq 2L}-1}$ is within a radius $2R$ of z_L, and the event $\mathcal{P}_L$ occurs.

We may therefore assume $\beta_2 \cdot \ell^\perp$ and $(\beta_1 - z'_L) \cdot \ell^\perp$ are both positive. This situation is depicted in Figure 3.2. If $\alpha_2 \cdot \ell^\perp > \beta_1 \cdot \ell^\perp$, then $\overline{\pi}^2$ must cross the line $\{x \cdot \ell^\perp = \beta_1 \cdot \ell^\perp\}$ at

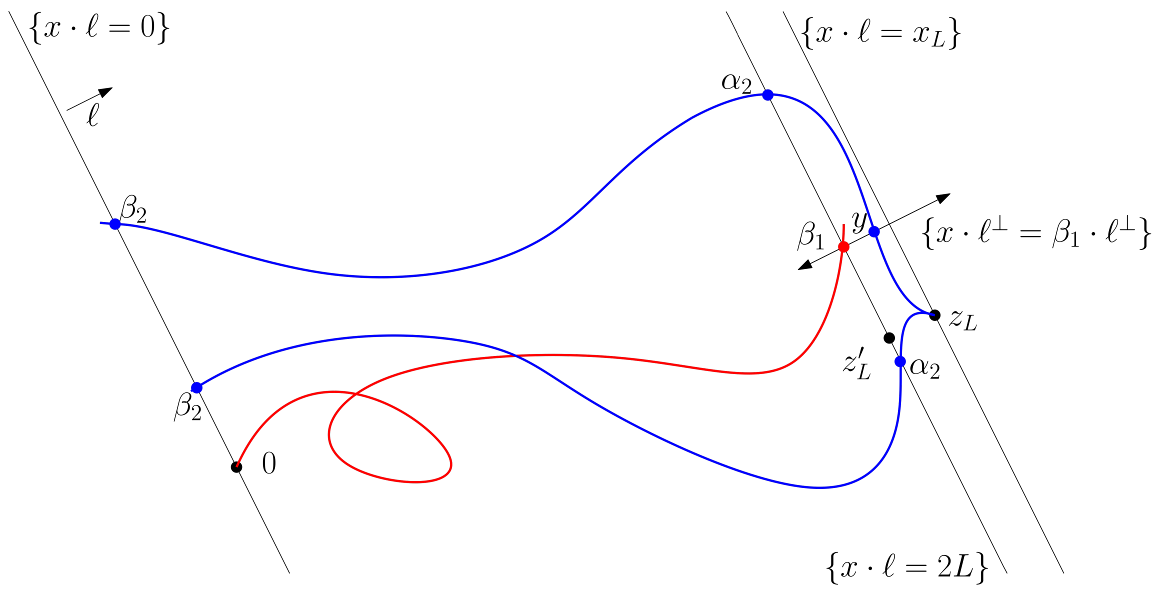

Figure 3.2. Two different ways to have $\beta_2 \cdot \ell^{\perp}$, $(\beta_1 - z'_L) \cdot \ell^{\perp} > 0$. The upper path from z_L shows the situation $\alpha_2 \cdot \ell^{\perp} > \beta_1 \cdot \ell^{\perp}$, while the lower path from z_L shows $\alpha_2 \cdot \ell^{\perp} < \beta_1 \cdot \ell^{\perp}$.

some point y between β_1 and $\beta_1 + (x_L - 2L)\ell$. This crossing point is a distance no more than $\frac{R}{2}$ from X^2_n for some $0 \leq n < T_{\leq 0}(\mathbf{X}^2)$. Its distance from β_1 is no more than R, and β_1 is no more than $\frac{R}{2}$ units of distance away from some X^1_m for some $0 \leq m \leq T_{\geq 2L}(\mathbf{X}^1)$. Thus, $\mathcal{P}_L$ occurs.

Finally, assume $\alpha_2 \cdot \ell^{\perp} < \beta_1 \cdot \ell^{\perp}$. Then the path taken by $\overline{\pi}^1$ from 0 to β_1 must intersect the path taken by $\overline{\pi}^2$ from α_2 to β_2, since they are paths connecting different pairs of opposite corners of the quadrilateral $(0, \beta_2, \beta_1, \alpha_2)$. The point of intersection is no more than $\frac{R}{2}$ units of distance away from X^1_m for some $0 \leq m \leq T_{\geq 2L}(\mathbf{X}^1)$ and no more than $\frac{R}{2}$ away from X^2_n for some $0 \leq n < T_{\leq 0}(\mathbf{X}^2)$. Thus, $\mathcal{P}_L$ occurs. This finishes the justification of (3.4), which together with (3.3) yields

$$\lim_{L \to \infty} \mathbb{P}^{0, z_L}(\mathcal{O}_L \cup \mathcal{P}_L) = \mathbb{P}^0(T_{<0} = \infty)\mathbb{P}^0(T_{>0} = \infty). \tag{3.5}$$

We are now interested in the event $\mathcal{O}_L \backslash \mathcal{P}_L$. Because this event does not involve the walks intersecting (and thus "sharing" part of the environment), its probability is the same as the probabilities of an analogous event where the two walks are run independently in *different* environments. And it is therefore bounded above by the probability of a similar event where two walks are run independently in different environments but are allowed to intersect paths. To formalize this idea, let $G_0^{2L,+}$ be the subset of G_0^{2L} on which $X_{T_{\geq 2L}} \cdot \ell^\perp > z_L$. Let $G_{x_L}^{0,+}$ be the subset of $G_{x_L}^0$ on which $X_{T_{\leq 0}} \cdot \ell^\perp > 0$. Define $G_0^{2L,-}$ and $G_{x_L}^{0,-}$ analogously. And define

$$\begin{aligned}
\Pi_L &:= \{(0 = X_0, X_1, \dots, X_{T_{\geq 2L}}) : \mathbf{X} \in G_0^{2L}\}, \\
\Pi_{L,+} &:= \{(0 = X_0, X_1, \dots, X_{T_{\geq 2L}}) : \mathbf{X} \in G_0^{2L,+}\}, \\
\Pi_{L,-} &:= \{(0 = X_0, X_1, \dots, X_{T_{\geq 2L}}) : \mathbf{X} \in G_0^{2L,-}\}.
\end{aligned}$$

We will abuse notation by using π to denote both a path in one of these sets and the set of vertices in that path. Then

$$\begin{aligned}
\mathbb{P}^{0,z_L}(\mathcal{O}_L \setminus \mathcal{I}_L) &= \sum_{\pi \in \Pi_{L,+}} \mathbb{P}^0(\mathbf{X} \text{ takes } \pi) \mathbb{P}^{z_L}(G_{x_L}^{0,-}, T_{\leq 0} < T_\pi) \\
&\qquad + \sum_{\pi \in \Pi_{L,-}} \mathbb{P}^0(\mathbf{X} \text{ takes } \pi) \mathbb{P}^{z_L}(G_{x_L}^{0,+}, T_{\leq 0} < T_\pi) \\
&\leq \sum_{\pi \in \Pi_{L,+}} \mathbb{P}^0(\mathbf{X} \text{ takes } \pi) \mathbb{P}^{z_L}(G_{x_L}^{0,-}) \\
&\qquad + \sum_{\pi \in \Pi_{L,-}} \mathbb{P}^0(\mathbf{X} \text{ takes } \pi) \mathbb{P}^{z_L}(G_{x_L}^{0,+}) \\
&= \mathbb{P}^0(G_0^{2L,+}) \mathbb{P}^{z_L}(G_{x_L}^{0,-}) + \mathbb{P}^0(G_0^{2L,-}) \mathbb{P}^{z_L}(G_{x_L}^{0,+}) \\
&\leq \frac{1}{2} \mathbb{P}^0(G_0^{2L}) \mathbb{P}^{z_L}(G_{x_L}^0). \\
&= \frac{1}{2} \mathbb{P}^0(G_0^{2L}) \mathbb{P}^0(G_0^{-x_L}). \\
&\underset{L \to \infty}{\longrightarrow} \frac{1}{2} \mathbb{P}^0(T_{<0} = \infty) \mathbb{P}^0(T_{>0} = \infty),
\end{aligned}$$

The last inequality comes from the median property of z_L, the last equality comes from translation invariance, and the limit comes from (3.2).

Hence, due to (3.5),

$$\frac{1}{2} \mathbb{P}^0(T_{<0} = \infty) \mathbb{P}^0(T_{>0} = \infty) \leq \liminf_{L \to \infty} \mathbb{P}^{0,z_L}(\mathcal{P}_L).$$

Therefore, to prove (3.1), it suffices to show

$$\lim_{L\to\infty} \mathbb{P}^{0,z_L}(\mathcal{P}_L \setminus \mathcal{I}_L) = 0. \tag{3.6}$$

and

$$\lim_{L\to\infty} \mathbb{P}^{0,z_L}(\mathcal{I}_L) = 0. \tag{3.7}$$

First, assume that

$$\mathbb{P}^{0,z_L}(\mathcal{P}'_L \setminus \mathcal{I}_L) \geq \frac{1}{2}\mathbb{P}^{0,z_L}(\mathcal{P}_L \setminus \mathcal{I}_L), \tag{3.8}$$

where $\mathcal{P}'_L \subset \mathcal{P}_L$ is the event $\mathbf{X}^1 \in G_0^{2L}$, $\mathbf{X}^2 \in G_{x_L}^0$, and $|X_m^1 - X_n^2| \leq 2R$ for some $0 \leq m \leq T_{\geq 2L}(\mathbf{X}^1)$, $0 \leq n < T_{\leq 0}(\mathbf{X}^2)$ with $X_m \cdot \ell \leq L$.

Now for a given path π, define the stopping time

$$T'_{\pi,L} := \inf\left\{n \geq 0 : \begin{array}{l}\text{for some } x \in \pi \text{ with } x \cdot \ell \leq L, \text{ there is a possible} \\ \text{path of length } M \text{ or less from } X_n \text{ to } x\end{array}\right\}.$$

Notice that $\mathcal{P}'_L$ implies that $T'_{\pi,L}(\mathbf{X}^2) \leq T_{\leq 0}(\mathbf{X}^2)$ for $\pi = (X_n^1)_{n=0}^{T_{\geq 2L}}$. This is because if $|X_m^1 - X_n^2| \leq 2R$, then there is a possible path of length no more than M from X_n^2 to X_m^1. Therefore,

$$\begin{aligned}
\mathbb{P}^{0,z_L}(\mathcal{P}'_L \setminus \mathcal{I}_L) &\leq \sum_{\pi\in\Pi} \mathbb{P}^{0,z_L}(\mathbf{X}^1 \text{ takes } \pi)\mathbb{P}^{0,z_L}(T'_{\pi,L}(\mathbf{X}^2) \leq T_{\leq 0}(\mathbf{X}^2) < T_\pi(\mathbf{X}^2) \wedge T_{>x_L}(\mathbf{X}^2)) \\
&= \sum_{\pi\in\Pi} \mathbb{P}^0(\mathbf{X} \text{ takes } \pi)\mathbb{P}^{z_L}(T'_{\pi,L} \leq T_{\leq 0} < T_\pi) \\
&\leq \sum_{\pi\in\Pi} \mathbb{P}^0(\mathbf{X} \text{ takes } \pi)\mathbb{P}^{z_L}(T'_{\pi,L} < \infty). \qquad (3.9)
\end{aligned}$$

The equality comes from independence of sites.

We next define another event for $(\mathbf{X}^1, \mathbf{X}^2)$ that involves the walks intersecting, but is not contained in the strip traversal event. Let the *meeting event* $\mathcal{M}_L$ be the event that $\mathbf{X}^1 \in G_0^{2L}$, and $\mathbf{X}^2$ intersects with the path $(X_n^1)_{n=0}^{T_{\geq 2L}}$ at a point y with $y \cdot \ell \leq L + RM$. Note that unlike the event $\mathcal{I}_L$, our event $\mathcal{M}_L$ does not require that $\mathbf{X}^2$ complete the event $G_{x_L}^0$, nor does it require that the intersection occur at some X_n^2 with $n \leq T_{\leq 0}(\mathbf{X}^2)$. However, unlike $\mathcal{I}_L$, $\mathcal{M}_L$ imposes the restriction that the intersection must occur on or near the half of the strip $\{0 \le x \cdot \ell \le 2L\}$ that is closer to 0.

Suppose $\mathbf{X}$ is such that $T'_{\pi,L} < \infty$ for a path $\pi \in \Pi$. Then there is a possible path $\gamma = (y_0 = X_{T'_{\pi,L}}, y_1, y_2, \ldots, y_k)$ with $k < M$, $y_k \in \pi$, and $y_k \cdot \ell \le L$. This path does not include any vertices in the path $(X_0, X_1, \ldots, X_{T'_{\pi,L}-1})$, because if such a vertex were included, there would be a shorter path from that vertex to y_k, violating the infimum part of the definition of $T'_{\pi,L}$. It may include multiple vertices from π, but taking $j = \inf\{0 \le i \le k : y_i \in \pi\}$, we may consider the path $\gamma' = (y_0 = X_{T'_{\pi,L}}, y_1, y_2, \ldots, y_j)$ that intersects neither $(X_0, X_1, \ldots, X_{T'_{\pi,L}-1})$ nor π, except at the terminating vertex. Therefore, by independence of sites,

$$\begin{aligned}\mathbb{P}^{z_L}\left((X_{T'_{\pi,L}+1}, X_{T'_{\pi,L}+2}, \ldots, X_{T'_{\pi,L}+j}) = \gamma' \,\middle|\, X_0, X_1, \ldots, X_{T'_{\pi,L}}\right) &= \mathbb{P}^{y_0}(\mathbf{X} \text{ takes } \gamma') \\ &\ge \kappa, \end{aligned} \tag{3.10}$$

where $\kappa > 0$ is the minimum annealed probability of any possible path of length less than M. A minimum exists because, up to translation invariance, there are only finitely many possible paths of a given length, and it is positive because by definition, all possible paths have positive annealed probability. Now y_j is of a distance at most $R(k-j) \le Rk < RM$ from y_k, and therefore $y_j \cdot \ell \le L + RM$. If, therefore, $\mathbf{X}^1$ takes π for some $\pi \in \Pi$, $T'_{\pi,L}(\mathbf{X}^2) < \infty$, and $(X^2_{T'_{\pi,L}+1}, X^2_{T'_{\pi,L}+2}, \ldots, X^2_{T'_{\pi,L}+j}) = \gamma'$, then $(\mathbf{X}^1, \mathbf{X}^2) \in \mathcal{M}_L$. Thus, since (3.10) is true whenever $T'_{\pi,L} < \infty$, we have

$$\mathbb{P}^{z_L}\left(T_{\pi \cap \{x \cdot \ell < L+RM\}} < \infty \,\middle|\, T'_{\pi,L} < \infty\right) \ge \kappa.$$

This gives us

$$\begin{aligned}\mathbb{P}^{0,z_L}(\mathcal{M}_L) &\ge \sum_{\pi \in \Pi} \mathbb{P}^0(\mathbf{X} \text{ takes } \pi)\mathbb{P}^{z_L}(T'_{\pi,L} < \infty)\kappa \\ &\overset{(3.9)}{\ge} \mathbb{P}^{0,z_L}(\mathcal{P}'_L)\kappa \\ &\overset{(3.8)}{\ge} \frac{1}{2}\kappa\mathbb{P}^{0,z_L}(\mathcal{P}_L), \end{aligned} \tag{3.11}$$

We will now show that $\mathbb{P}^{0,z_L}(\mathcal{M}_L)$ vanishes. Zerner shows in [31] that $\mathcal{I}_L$ has vanishing probability. The argument also works for our event $\mathcal{M}_L$. We summarize it here, applying it to $\mathcal{M}_L$. Fix $\varepsilon > 0$, and suppose the intersection occurs at a point y. Either $P_\omega^y(A_\ell) < \varepsilon$ or $P_\omega^y(A_\ell) \geq \varepsilon$. In the former case, a walk from 0 passes through y but still has $T_{\geq L} < T_{<0}$. Zerner shows in [31] that the probability of this event has limsup bounded above by ε. In the latter case, a walk started from z_L travels a great distance in direction $-\ell$ (here, a distance at least $L - RM$) and still reaches a point where the probability of A_ℓ is at least ε. The chance of traveling such a distance in direction $-\ell$ but still having $\mathbf{X}^2 \in A_\ell$ approaches 0 as $L \to \infty$. On the other hand, if $\mathbf{X}^2 \in A_{-\ell}$, then $P_\omega^{X_n}(A_\ell)$ must approach 0, being a bounded martingale, and so the probability that it is still above ε after $\frac{L-RM}{R}$ units of time (long enough to travel distance $L - RM$) approaches 0 as $L \to \infty$. One may then take ε to 0. Hence we may conclude that

$$\lim_{L\to\infty} \mathbb{P}^{0,z_L}(\mathcal{M}_L) = 0.$$

Since (3.11) is true whenever (3.8) is true, we may conclude that

$$\lim_{L\to\infty} \mathbb{P}^{0,z_L}(\mathcal{M}_L)\mathbb{1}_{(3.8)\text{ holds}} = 0.$$

By a nearly symmetric argument,[2]

$$\lim_{L\to\infty} \mathbb{P}^{0,z_L}(\mathcal{M}_L)\mathbb{1}_{(3.8)\text{ does not hold}} = 0.$$

It follows that $\lim_{L\to\infty} \mathbb{P}^{0,z_L}(\mathcal{P}_L) = 0$, which is (3.6). To get (3.7), note that the subset of $\mathcal{I}_L$ where an intersection occurs to the left of the line $\{x \cdot \ell = L + RM\}$ is actually contained in $\mathcal{M}_L$, and therefore its probability vanishes with that of $\mathcal{M}_L$. The subset where an intersection occurs to the right of $\{x \cdot \ell = L + RM\}$ vanishes by a nearly symmetric consideration. □

[2]↑It is not perfectly symmetric because of the possible difference between x_L and $2L$, but the logic is unchanged.

3.2 Directional transience

Our next goal is to extend [15, Theorem 1] to bounded jumps, proving Theorem 2.2.1, which we recall here.

Theorem (Theorem 2.2.1)**.** *Let $\mathbb{P}^0_{\mathcal{G}}$ be the measure of a RWDE with bounded jumps on $\mathbb{Z}^d$. Let $\Delta = \mathbb{E}^0[X_1]$ be the annealed drift, and let $\ell \in S^{d-1}$. Then $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 1$ if and only if $\ell \cdot \Delta > 0$; otherwise, $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 0$.*

As we've discussed, the first part (regarding nonzero annealed drift) was proven for general RWDE on $\mathbb{Z}^d$ with bounded jumps in [22]. However, we outline the argument in one dimension here, showing that if $\kappa_1 \neq 0$, then the walk is almost surely transient in the direction of κ_1. This, combined with our argument for recurrence in the zero-drift case (which uses a graph similar to the one used in [22]), should give the reader a good idea of how the directional transience result is proven. We also include the summary in order to emphasize the reason why κ_1 plays a role in characterizing ballisticity as well as directional transience. The finite graphs $\mathcal{G}_M$ used in the transience proof are closely related to a one-sided infinite "limiting graph" $\mathcal{G}_+$ used in the ballisticity proofs, and comparisons between $\mathcal{G}_M$ and $\mathcal{G}_+$ can be used to prove a crucial lemma about $\mathcal{G}_+$. Including the argument for transience now allows this connection to be seen. Finally, we include it because the ideas in the proof of Claim 3.2.1.1 are used repeatedly in Chapter 4, and this is a natural place to present the thorough version of the argument for later reference.

Theorem 3.2.1.

If $\kappa_1 > 0$, then $\lim_{n\to\infty} X_n = \infty$*,* $\mathbb{P}^0_{\mathcal{G}}$*–a.s..*

Proof (outline). This follows from [22, Corollary 1], but we outline the proof for the one-dimensional case. We are to show that $\lim_{n\to\infty} \frac{X_n}{n} = \infty$, $\mathbb{P}^0_{\mathcal{G}}$–a.s. The steps are as follows:

(a) Show that $\mathbb{P}^0_{\mathcal{G}}(T_{\geq M} < \tilde{T}_{\leq 0})$ is bounded away from 0 as M approaches ∞. (This is shown in [22, Theorem 1]).

(b) Taking limits, conclude that $\mathbb{P}^0_{\mathcal{G}}(\tilde{T}_{\leq 0} = \infty) > 0$, implying the walk is transient to the right with positive probability.

(c) Use the 0-1 law [16, Theorem 11] to conclude that if the probability of transience to the right is positive, it is 1.

We outline the proof of (a) for convenient reference. For $M > R + L$, we consider a weighted directed graph $\mathcal{G}_M$ with vertex set $[0, M]$. Each vertex in $[1, M-1]$ has the same edges with the same weights as those on $\mathcal{G}$, except that edges that would terminate at points less than zero are simply edges to the point 0, and edges that would terminate at points greater than M are simply edges to the point M. If this would result in multi-edges, each multi-edge is replaced with a single edge whose weight is the sum of the weights in the multi-edge; however, we leave the multi-edges in our illustrations in order to show more clearly where this occurs. Based on the edges and weights we've described so far, zero divergence already holds at points from $R+1$ to $M-L-1$, but points to the left of R and to the right of $M-L$ are "missing" incoming weights from vertices to the left of 0 and to the right of M, respectively. Therefore, to each vertex $1 \le j \le R$, we add an edge from 0 with weight $\sum_{i=j}^{R} \alpha_i$ (depicted in Figure 3.3 as a multi-edge) in order to achieve zero divergence at j. Likewise, to each edge $M-L \le j \le M$, we add an edge from M with weight $\sum_{i=M-L}^{M-j} \alpha_{i-M}$. Based on these weights, the site 0 has incoming weight d^- and outgoing weight d^+, and the site M has incoming weight d^+ and outgoing weight d^-. To adjust for this, we add a special edge from M to 0 with weight $d^+ - d^- = \kappa_1$.

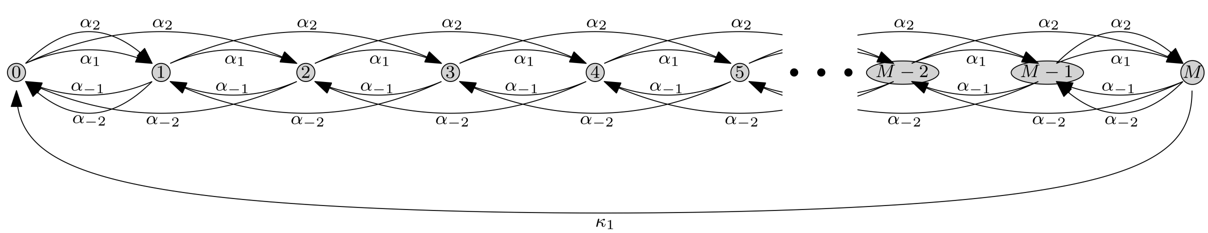

Figure 3.3. The graph $\mathcal{G}_M$.

Now the graph satisfies the zero-divergence property. This is the graph $\mathcal{G}_M$, pictured in Figure 3.3. For now, the usefulness of $\mathcal{G}_M$ comes from the following claim, which we prove in its entirety because the ideas in its proof will be referenced several times throughout this paper.

Claim 3.2.1.1.

$$\mathbb{P}^x_{\mathcal{G}}(T_{\geq M} < T_{\leq 0}) = \mathbb{P}^x_{\mathcal{G}_M}(T_M < T_0), \qquad 0 < x < M. \tag{3.12}$$

To prove this, consider for each environment ω on $\mathbb{Z}$ a modified environment ω', where transition probabilities between sites in $[1, M-1]$ are the same as in ω, but for each $i \in [M-R, M-1]$, $\omega'(i,M) = \sum_{j \geq M} \omega(i,j)$, and for each $i \in [1,R]$, $\omega'(i,0) = \sum_{j \leq 0} \omega(i,j)$. Then by construction, a walk drawn according to $P^x_{\omega'}$ for any x strictly between 0 and M and stopped when it hits 0 or M follows the same law (except possibly for the terminating site) as the law of a walk drawn according to P^x_ω and stopped when it reaches $(-\infty, 0]$ or $[M, \infty)$. In particular,

$$P^x_\omega(T_{\geq M} < T_{\leq 0}) = P^x_{\omega'}(T_M < T_0), \qquad 0 < x < M.$$

On the other hand, by the amalgamation property of Dirichlet random vectors, we also see that for every $y \in [1, M-1]$, the law of $(\omega')^x$ under $P_{\mathcal{G}}$ is a Dirichlet distribution, and in fact is the same as the law of ω^x under $P_{\mathcal{G}_M}$. Hence, for each $0 < x < M$, we have

$$\begin{aligned}
\mathbb{P}^x_{\mathcal{G}}(T_{\geq M} < T_{\leq 0}) &= E_{\mathcal{G}}\left[P^x_\omega(T_{\geq M} < T_{\leq 0})\right] \\
&= E_{\mathcal{G}}\left[P^x_{\omega'}(T_M < T_0)\right] \\
&= E_{\mathcal{G}_M}\left[P^x_\omega(T_M < T_0)\right] \\
&= \mathbb{P}^x_{\mathcal{G}_M}(T_M < T_0).
\end{aligned}$$

This proves the claim.

From Corollary 1.4.3 (2), we can get $\mathbb{P}^0_{\mathcal{G}_M}(T_M < \tilde{T}_0) \geq \frac{\kappa_1}{d^+}$. We can use Claim 3.2.1.1 to show that $\mathbb{P}^0_{\mathcal{G}_M}(T_M < \tilde{T}_0) \leq \mathbb{P}^R_{\mathcal{G}}(T_{\geq M} < T_{\leq 0})$. Putting the two together, we get

$$\mathbb{P}^R_{\mathcal{G}}(T_{\geq M} < T_{\leq 0}) \geq \frac{\kappa_1}{d^+}$$

for all M. By independence of sites, we then have

$$\begin{aligned}\mathbb{P}^0_{\mathcal{G}}(T_{\geq M} < \tilde{T}_{\leq 0}) &\geq \mathbb{P}_{\mathcal{G}}(X_1 = R)\mathbb{P}^R_{\mathcal{G}}(T_{\geq M} < T_{\leq 0}) \\ &\geq \mathbb{P}_{\mathcal{G}}(X_1 = R)\frac{\kappa_1}{d^+}.\end{aligned}$$

This is the bound we needed. □

3.3 Directional recurrence

To complete the proof of Theorem 2.2.1, all that remains is to prove the following.

Theorem (Theorem 2.2.2)**.** *Let $\mathbb{P}^0_{\mathcal{G}}$ be the measure of a RWDE with bounded jumps on $\mathbb{Z}^d$. Let $\Delta = 0$, and let $\ell \in S^{d-1}$. Then $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 0$.*

To make our arguments easier to follow, we will first consider directions ℓ with rational slopes. Let $S^{d-1}_r := \left\{\frac{u}{|u|} : u \in \mathbb{Z}^d \setminus \{0\}\right\} \subset S^{d-1}$ be the set of vectors in the unit sphere S^{d-1} that have all rational slopes; we will prove Theorem 2.2.2 for vectors in this set first. However, because we cannot rely on the truth of Conjecture 2.2.3, proving Theorem 2.2.2 for all directions $\ell \in S^{d-1}_r$ is not sufficient to prove it for all directions $\ell \in S^{d-1}$, even though S^{d-1}_r is dense in S^{d-1}. For $\ell \in S^{d-1} \setminus S^{d-1}_r$, we must rule out the possibility that a walk could with positive probability be transient in direction ℓ while recurrent in all directions not parallel to ℓ.

Once we prove Theorem 2.2.2 for rational slopes, we do not know of a way to directly generalize to arbitrary directions. The generalization will require going through the same argument more carefully, choosing directions $v \in S^{d-1}_r$ sufficiently close to ℓ to satisfy certain properties. However, to make our arguments easier to follow, we first prove the theorem for directions $\ell \in S^{d-1}_r$, and then afterward describe the differences necessary for dealing with directions $\ell \in S^{d-1} \setminus S^{d-1}_r$. Because we are discussing multiple directions, we must replace our simplified notation $T_{\diamond a}$ for hitting times of half-spaces with the slightly more cumbersome $T^\ell_{\diamond a}$. In order to facilitate comparisons later, we use this notation even for the simplified version of the proof that assumes $\ell \in S^{d-1}_r$ and only discusses one direction.

Moreover, for $\ell \in S^{d-1}$ we will also need to define the "lateral hitting times"

$$H^{\ell}_{\geq a} := \inf\{n \geq 0 : X_n \cdot \ell^{\perp} \geq a \text{ for some } \ell^{\perp} \perp \ell\}.$$

We are now ready to prove our Theorem 2.2.2 for directions with rational slopes.

3.3.1 Rational slopes

Theorem 3.3.1. *Let $\mathbb{P}^0_{\mathcal{G}}$ be the measure of a RWDE with bounded jumps on $\mathbb{Z}^d$. Let $\Delta = 0$, and let $\ell \in S^{d-1}_r$. Then $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 0$.*

Proof of Theorem 3.3.1. Let $\ell \in S^{d-1}_r$. Assume for a contradiction that $\mathbb{P}^0_{\mathcal{G}}(A_{-\ell}) > 0$. Then, as in [30, page 765], we have $\mathbb{P}^0_{\mathcal{G}}(T^{\ell}_{>0} = \infty) > 0$, from which it easily follows that $\alpha := \mathbb{P}^0_{\mathcal{G}}(\tilde{T}^{\ell}_{\geq 0} = \infty) > 0$.

Lemma 4 from [23] is stated for nearest-neighbor RWRE, but its proof can easily be modified to work for RWRE satisfying our assumptions. It says it is $\mathbb{P}^0$–a.s. impossible to visit a slab of finite width infinitely often without visiting both of its neighboring half-spaces, and it implies that on the event $\{\tilde{T}^{\ell}_{\geq 0} = \infty\}$, it is also the case (up to a set of probability zero) that $T^{\ell}_{\leq -L} < \tilde{T}^{\ell}_{\geq 0}$. Since it is always possible, with positive probability, for a walk to hit the half-space $\{x \cdot \ell \leq -L\}$ and then return to $\{x \cdot \ell \geq 0\}$, we have $\mathbb{P}^0_{\mathcal{G}}(T^{\ell}_{\leq -L} < \tilde{T}^{\ell}_{\geq 0}) > \alpha$ for all $L \geq 0$. And on the event $\{T^{\ell}_{\leq -L} < \tilde{T}^{\ell}_{\geq 0}\}$, there is necessarily some a such that $T^{\ell}_{\leq -L} < \tilde{T}^{\ell}_{\geq 0} \wedge H^{\ell}_{\geq a}$. It therefore follows that for any $L > 0$, there exists $K = K(L) > 0$ such that

$$\mathbb{P}^0_{\mathcal{G}}(T^{\ell}_{\leq -L} < \tilde{T}^{\ell}_{\geq 0} \wedge H^{\ell}_{\geq \frac{1}{2}K}) > \alpha. \tag{3.13}$$

For $L \geq R$, let $K = K(L)$ be an increasing function satisfying (3.13) for all L. Let u be a constant multiple of ℓ such that $u \in \mathbb{Z}^d$. Then let $(u, u_2, \ldots, u_d)$ be an orthogonal basis for $\mathbb{R}^d$ such that $u_i \in \mathbb{Z}^d$ for all i. Let N be large enough that $N|u_i| \geq K$ for all i.

We will define a graph $\mathcal{H}_{N,L}$ in nearly the same way as the $G_{N,L}$ defined by Tournier in [22]. Consider the cylinder

$$C_{N,L} := \{x \in \mathbb{Z}^d : 0 \leq x \cdot \ell \leq L\}/(N\mathbb{Z}u_2, \ldots, N\mathbb{Z}u_d).$$

This is the slab $\mathcal{S}_{N,L} := \{0 \le x \cdot \ell \le L\} \cap \mathbb{Z}^d$ where vertices that differ by Nu_i for some $i \in \{2, \dots, d\}$ are identified. We note that [22] uses $L|u|$ rather than L here. We use L for reasons related to our plans for generalizing the proof to $\ell \notin S_r^{d-1}$.

Now define the graph $\mathcal{H}_{N,L}$ with vertex set

$$V_{N,L} := C_{N,L} \cup \{M\} \cup \{\partial\},$$

where M and ∂ are new vertices (in [22], R and ∂ are used, but in this paper, R already has a meaning), and edges of $\mathcal{H}_{N,L}$ are of the following types:

1. edges induced by those of $\mathcal{G}$ inside $C_{N,L}$;
2. If $x \in C_{N,L}$ corresponds to a vertex $x' \in \mathcal{S}_{N,L}$, there is
   (a) an edge from x to ∂ for each $y \in \mathcal{N}$ such that $(x' + y) \cdot \ell < 0$,
   (b) an edge from ∂ to x for each $y \in -\mathcal{N}$ such that $(x' + y) \cdot \ell < 0$,
   (c) an edge from x to M for each $y \in \mathcal{N}$ such that $(x' + y) \cdot \ell > L$,
   (d) an edge from M to x for each $y \in -\mathcal{N}$ such that $(x' + y) \cdot \ell > L$,
3. A new "special" edge from M to ∂ and one from ∂ to M.

Weights of all edges but the last two are induced by the corresponding weights in $\mathcal{G}$. Note that several edges may share the same head and tail. If that is the case, identify such edges into one edge whose weight is the sum of all of the original weights in order to create a graph that is not a multigraph and fits our definitions (there is also a way to define RWDE on a multigraph by keeping track of vertices visited and edges taken, and if we used such a definition, the identification of multiple edges would not affect the distribution of the vertex path). By construction, zero divergence holds at all vertices in $C_{N,L}$. It remains to describe the weights of the new edges connecting ∂ and M (the paper [22] only defines an edge from M to ∂). The paper [22] shows that the quantity

$$\Big(\sum \text{weights of edges in 2(c)}\Big) - \Big(\sum \text{weights of edges in 2(a)}\Big)$$

is a multiple of the annealed drift. Thus, because of our assumption the annealed drift is zero, the two sums are equal. Note that by the shift-invariant structure of the graph $\mathcal{G}$, the sum on the left is also the weight exiting ∂ by edges in 2(b). Similarly, the sum on the right is also the weight exiting M by edges in 2(d). Hence the total weights of edges in 2(a), 2(b), 2(c), and 2(d) are all the same. Because weights in 2(a) and 2(b) are the same, zero divergence holds at ∂, and because 2(c) and 2(d) are the same, zero divergence holds at M. In order to preserve zero divergence, we give both of the special edges the same weight W, which we take to be the value of each of the two sums above. It follows from well known properties of Dirichlet random variables that when the walk is started at either of the endpoints, its first step is along the special edge to the other endpoint with annealed probability $\frac{1}{2}$. Figure 3.4 shows an example of the graph $\mathcal{H}_{N,L}$. (Because Figure 3.4 is also intended to be used for the argument for Theorem 2.2.2, it uses v rather than ℓ in its labeling. For the purpose of the current argument, simply take $v = \ell$.)

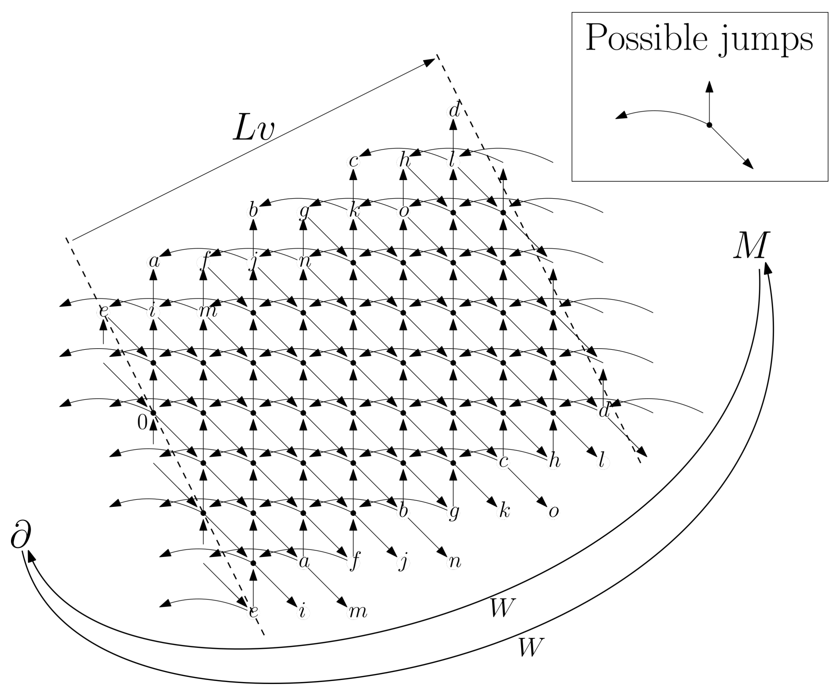

Figure 3.4. Graph $\mathcal{H}_{N,L}$. Here $\mathcal{N} = \{(0,1), (1,-1), (-2,0)\}$, and $v = (2,1)$. Boundary conditions in direction perpendicular to v are periodic; vertices labeled with the same letters are identified. Arrows to and from the main part of the graph on the left are understood to originate from or terminate at ∂, and similarly with M on the right side.

Define the stopping time $\tau = \inf\{n \in \mathbb{N}_0 : X_n = 0, X_{n-1} = M\}$. Note that $\{\tilde{T}_\partial = \tau\} = \{X_{\tilde{T}_\partial - 1} = M\}$ is the event that the first return to zero is by the special edge.

We note, by Lemma 1.4.3,

$$\mathbb{P}^\partial_{\mathcal{H}_{N,L}}(\tilde{T}_\partial = \tau) = \frac{1}{2}$$

On the other hand, we also have $\mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 = M) = \frac{1}{2}$. Now by considering the possibility that the first step from ∂ is to M (by the special edge) and the possibility that the first step from ∂ is not to M, we get

$$\mathbb{P}^\partial_{\mathcal{H}_{N,L}}(\tilde{T}_\partial = \tau) \leq \mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 = M)\mathbb{P}^M_{\mathcal{H}_{N,L}}(T_\partial = \tau) + \mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 \neq M, T_M < \tilde{T}_\partial),$$

which can be rewritten as

$$\frac{1}{2} \leq \frac{1}{2}\mathbb{P}^M_{\mathcal{H}_{N,L}}(\tilde{T}_\partial = \tau) + \mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 \neq M, T_M < \tilde{T}_\partial), \tag{3.14}$$

We claim that the term $\mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 \neq M, T_M < \tilde{T}_\partial)$ approaches 0 as L and K increase. Let $B = B(L, K)$ be a box of radius $\frac{L \wedge K}{3}$ around 0, and for $x \in C_{N,L}$, let $x + B$ be the set of vertices in $C_{N,L}$ that can be written as $x + y$ for some $y \in B$. Note that for $x \in C_{N,L}$, the dot product with ℓ is well defined, since vertices in $\mathcal{S}_{N,L}$ that are identified to form $C_{N,L}$ have the same dot product with ℓ. Then for sufficiently large L,

$$\begin{aligned}
\mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 \neq M, T_M < \tilde{T}_\partial) &= \sum_{\substack{x \in C_{N,L}, \\ 0 \leq x\cdot\ell \leq R}} \mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 = x)\mathbb{P}^x_{\mathcal{H}_{N,L}}(T_M < T_\partial) \\
&\leq \sum_{\substack{x \in C_{N,L}, \\ 0 \leq x\cdot\ell \leq R}} \mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 = x)\mathbb{P}^x_{\mathcal{H}_{N,L}}(T_{(x+B)^c} < T_\partial) \\
&\leq \sum_{\substack{x \in C_{N,L}, \\ 0 \leq x\cdot\ell \leq R}} \mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 = x)\mathbb{P}^0_{\mathcal{G}}(T_{B^c} < T^\ell_{\leq -R}) \\
&= \mathbb{P}^0_{\mathcal{G}}(T_{B^c} < T^\ell_{\leq -R})
\end{aligned} \tag{3.15}$$

The first equality comes from the strong Markov property and independence of sites. The first inequality holds as long as L is large enough that $M \notin x + B$. To get the second inequality, note that a finite path from x that stays in $x+B$ until the last step does not use the periodic boundary conditions (provided $\frac{L\wedge K}{3} > R$), and so it has the same probability as a corresponding path in $\mathcal{G}$. And for $x \in C_{N,L}$ with $x \cdot \ell \le R$, a walk from x on $\mathcal{H}_{N,L}$ that leaves $x+B$ without hitting ∂ corresponds to a walk on $\mathcal{G}$ (which we may take to start at 0 by translation invariance) that leaves B without traveling $x \cdot \ell$ or more units (of distance in $\mathbb{R}^d$) in direction $-\ell$. Since $x \cdot \ell \le R$ for all x with $\mathbb{P}^{\partial}_{\mathcal{H}_{N,L}}(X_1 = x) > 0$, the second inequality follows. The final equality comes from pulling the second term out of the sum, which is then equal to 1.

To prove our claim, we must show that (3.15) goes to 0 as L increases (along with K). Let $\varepsilon > 0$. By assumption, $\mathbb{P}^0_{\mathcal{G}}(A_{-\ell}) > 0$. By Theorem 2.1.1 for $d = 2$, or by the 0-1 law of Bouchet for $d \ge 3$ in [14] (where, as Tourner points out in [22], the proof works for bounded jumps), this means $\mathbb{P}^0_{\mathcal{G}}(A_{-\ell}) = 1$. Thus, $\mathbb{P}^0_{\mathcal{G}}(T^{\ell}_{\le -R} < \infty) = 1$. Now take an increasing sequence (Q_r) of finite sets converging to $\mathbb{Z}^d$. Then the event $\{T^{\ell}_{\le -R} < \infty\}$ is the limit as r increases (i.e., the union over all r) of the events $\{T^{\ell}_{\le -R} < T_{Q_r^c}\}$. Let $Q = Q(\varepsilon)$ be one such Q_r large enough that $\mathbb{P}^0_{\mathcal{G}}(T_{Q^c} < T^{\ell}_{\le -R}) < \varepsilon$. Note that although Q depends on ε, it does not depend on L. Thus, for large enough L, B contains Q, so that

$$\{T_{B^c} \le T^{\ell}_{\le -R}\} \subset \{T_{Q^c} \le T^{\ell}_{\le -R}\}. \tag{3.16}$$

It follows that, for large enough L,

$$\mathbb{P}^0_{\mathcal{G}}(T_{B^c} \le T^{\ell}_{\le -R}) \le \mathbb{P}^0_{\mathcal{G}}(T_{Q^c} \le T^{\ell}_{\le -R}) < \varepsilon.$$

Since this can be true for arbitrary $\varepsilon > 0$, the right side of (3.15) goes to 0, and therefore so does $\mathbb{P}^{\partial}_{\mathcal{H}_{N,L}}(X_1 \ne M, T_M < \tilde{T}_{\partial})$.

Next, we will show that $\mathbb{P}^{M}_{\mathcal{H}_{N,L}}(T_{\partial} = \tau)$ is bounded away from 1 as M increases. We have

$$
\begin{aligned}
\mathbb{P}^M_{\mathcal{H}_{N,L}}(T_\partial \neq \tau) &\geq \sum_{x \in C_{N,L}, L-R \leq x\cdot\ell \leq L} \mathbb{P}^M_{\mathcal{H}_{N,L}}(X_1 = x)\mathbb{P}^x_{\mathcal{H}_{N,L}}(T_\partial < T^\ell_{>x\cdot\ell}) \\
&\geq \sum_{x \in C_{N,L}, L-R \leq x\cdot\ell \leq L} \mathbb{P}^M_{\mathcal{H}_{N,L}}(X_1 = x)\mathbb{P}^0_{\mathcal{G}}(T^\ell_{\leq -L} < \tilde{T}^\ell_{\geq 0} \wedge H^\ell_{\geq \frac{1}{2}K}) \\
&> \sum_{x \in C_{N,L}, L-R \leq x\cdot\ell \leq L} \mathbb{P}^M_{\mathcal{H}_{N,L}}(X_1 = x)\alpha \\
&= \frac{1}{2}\alpha.
\end{aligned}
$$

The first inequality comes from the strong Markov property and independence of sites. To get the second inequality, note that the probability $\mathbb{P}^x_{\mathcal{H}_{N,L}}(T_\partial < T_{>x\cdot\ell})$ is greater than the probability, starting from x, that a walk on $\mathcal{H}_{N,L}$ reaches ∂ without ever traveling more than $\frac{N}{3}$ units in any direction perpendicular to u. Since this event precludes the walk from using the periodic boundary conditions, (and because weights to ∂ in $\mathcal{H}_{N,L}$ are the same as the weights from corresponding sites to the set $\{y : y \cdot u < 0\}$) its probability is the same as the probability that a walk in $\mathcal{G}$ travels more than $x \cdot \ell$ units in direction $-u$ without ever traveling more than $\frac{N}{3}$ units in any perpendicular direction. Since $x \cdot u \leq L$, the second inequality follows. The third inequality comes from (3.13), and the equality comes from the expectation of a beta random variable.

Now taking the limsup in (3.14) as $M \to \infty$ yields the contradiction

$$\frac{1}{2} \leq \frac{1}{2}\left(1 - \frac{1}{2}\alpha\right) < \frac{1}{2}.$$

□

3.3.2 Generalizing to directions in $S^{d-1} \setminus S^{d-1}_r$

We now describe how to prove Theorem 3.3.1 for directions that do not necessarily have rational slopes.

The graph constructed in [22] is used to analyze a direction ℓ with rational slopes, and uses the rationality in a significant way. Rather than attempt to construct and analyze an analogous graph for an irrational direction $\ell \in S^{d-1}$, we use a sequence of rational slopes $v \in S_r^{d-1}$ approaching ℓ. The following lemma is simple, but important.

Lemma 3.3.2. *Fix $\ell \in S^{d-1}$, $h > 0$, and $L' > L > 0$. For v close enough to ℓ, any $x \in \mathbb{R}^d$ with $x \cdot \ell \geq L'$ and $x \cdot v \leq L$ must necessarily have $x \cdot \ell^\perp > h$ for some $\ell^\perp \perp \ell$.*

Proof. Choose a unit vector v close to ℓ and let $\ell^\perp \in S^{d-1}$ be the unit vector perpendicular to ℓ such that $v = a\ell - \sqrt{1-a^2}\ell^\perp$, where $a = v \cdot \ell$. Then $a \nearrow 1$ as $v \to \ell$, and $\ell - v = (1-a)\ell + \sqrt{1-a^2}\ell^\perp$. By writing $x \cdot (\ell - v)$ in different ways, we get

$$(1-a)x \cdot \ell + \sqrt{1-a^2}x \cdot \ell^\perp = x \cdot \ell - x \cdot v.$$

From this we get

$$\begin{aligned} \sqrt{1-a^2}x \cdot \ell^\perp &= ax \cdot \ell - x \cdot v \\ &\geq aL' - L. \end{aligned}$$

For v sufficiently close to ℓ, a is close enough to 1 that this gives us

$$\sqrt{1-a^2}x \cdot \ell^\perp \geq \frac{1}{2}(L' - L)$$

and

$$x \cdot \ell^\perp \geq \frac{1}{2\sqrt{1-a^2}}(L' - L).$$

Taking v close to ℓ makes a close to 1, which suffices to prove the lemma. $\square$

We now proceed with the proof, describing only the parts where it differs from the proof of Theorem 3.3.1.

Proof of Theorem 2.2.2. The first challenge is to get the same bound as in (3.13), but for a direction v with rational slopes. We will show that for any L, there is a unit vector $v = v(L) \in S_r^{d-1}$ close enough to ℓ and a $K = K(L)$ large enough that

$$\mathbb{P}_{\mathcal{G}}^0(T_{\leq -L}^v < \tilde{T}_{\geq 0}^v \wedge H_{\geq \frac{1}{2}K}^v) > \alpha. \tag{3.17}$$

Fix $L > 0$, and choose any $L' > L$. Let K' be such that

$$\mathbb{P}_{\mathcal{G}}^{0}(T^{\ell}_{\leq -L'} < \tilde{T}^{\ell}_{\geq 0} \wedge H^{\ell}_{\geq \frac{1}{2}K'}) > \alpha.$$

(Such a K' exists by (3.13).) Now on the event $\{T^{\ell}_{\leq -L'} < \tilde{T}^{\ell}_{\geq 0}\}$, there is necessarily an open neighborhood around ℓ such that for any v in the neighborhood, $T^{\ell}_{\leq -L'} < \tilde{T}^{v}_{\geq 0}$. This is because the walk only hits finitely many points before $T_{\leq -L'}$, and each such point x (other than 0) has $x \cdot \ell < 0$, so that for v close enough to ℓ, $x \cdot v < 0$. Hence

$$\lim_{v \to \ell} \mathbb{P}_{\mathcal{G}}^{0}(T^{\ell}_{\leq -L'} < \tilde{T}^{\ell}_{\geq 0} \wedge \tilde{T}^{v}_{\geq 0} \wedge H^{\ell}_{\geq \frac{1}{2}K'}) = \mathbb{P}_{\mathcal{G}}^{0}(T^{\ell}_{\leq -L'} < \tilde{T}^{\ell}_{\geq 0} \wedge H^{\ell}_{\geq \frac{1}{2}K'}).$$

In particular, for v close enough to ℓ,

$$\mathbb{P}_{\mathcal{G}}^{0}(T^{\ell}_{\leq -L'} < \tilde{T}^{\ell}_{\geq 0} \wedge \tilde{T}^{v}_{\geq 0} \wedge H^{\ell}_{\geq \frac{1}{2}K'}) > \alpha. \tag{3.18}$$

Now let $v \in S^{d-1}$ have rational slopes, satisfy (3.18), and also be close enough to ℓ that if $x \cdot \ell \geq L'$ and $x \cdot v \leq L$, then $x \cdot \ell^{\perp} \geq K'$ for some $\ell^{\perp} \perp \ell$ (this is possible by Lemma 3.3.2). Choose K large enough that any y with $y \cdot v^{\perp} \geq \frac{K}{2}$ for any $v^{\perp} \perp v$ is necessarily outside the set

$$Z := \left\{ -L' - R \leq x \cdot \ell \leq 0, x \cdot v \leq 0, x \cdot \ell^{\perp} \leq \frac{K'}{2} \text{ for all } \ell^{\perp} \perp \ell \right\}.$$

See Figure 3.5. Now on the event $\{T^{\ell}_{\leq -L'} < \tilde{T}^{\ell}_{\geq 0} \wedge \tilde{T}^{v}_{\geq 0} \wedge H^{\ell}_{\geq \frac{1}{2}K'}\}$, it is necessarily the case that $X_n \in Z$ for $0 \leq n \leq T^{\ell}_{\leq -L'}$. Furthermore, if $z := X_{T^{\ell}_{\leq -L'}}$, then since $z \cdot \ell \leq -L'$ and $z \cdot \ell^{\perp} \leq \frac{K'}{2}$ for all $\ell^{\perp} \perp \ell$, the choice of v implies that $z \cdot v \leq -L$. Thus $X_n \in Z$ for $0 \leq n \leq T^{v}_{\leq -L}$, and therefore $T^{v}_{\leq -L} < \tilde{T}^{v}_{\geq 0} \wedge H^{v}_{\geq \frac{1}{2}K}$. Hence (using (3.18)),

$$\mathbb{P}_{\mathcal{G}}^{0}(T^{v}_{\leq -L} < \tilde{T}^{v}_{\geq 0} \wedge H^{v}_{\geq \frac{1}{2}K}) \geq \mathbb{P}_{\mathcal{G}}^{0}(T^{\ell}_{\leq -L'} < \tilde{T}^{\ell}_{\geq 0} \wedge \tilde{T}^{v}_{\geq 0} \wedge H^{\ell}_{\geq \frac{1}{2}K'}) > \alpha.$$

This is (3.17).

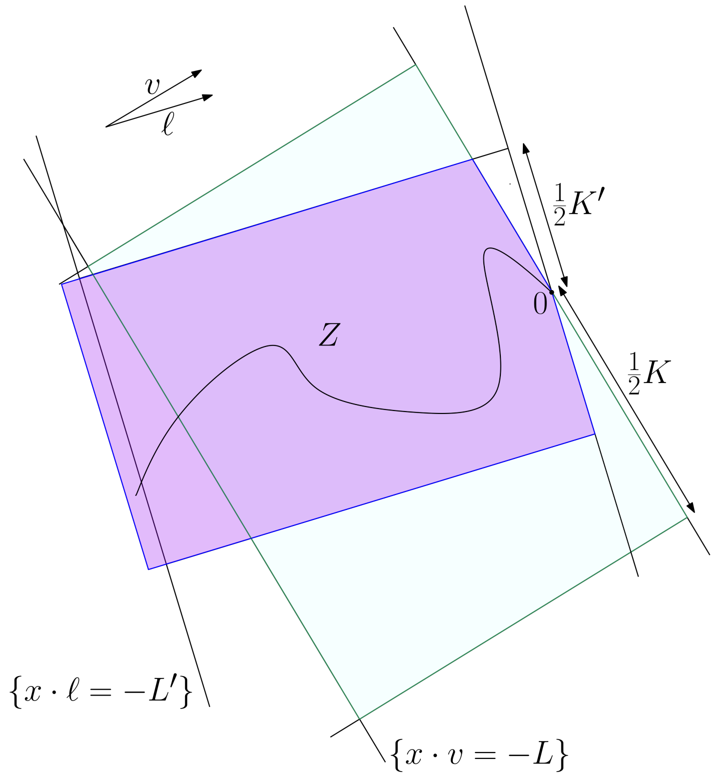

Figure 3.5. In order for the walk to cross the line $\{x \cdot \ell = -L'\}$ before leaving the set Z, it must exit the lighter shaded box through the line $\{x \cdot v = -L\}$.

For $L \geq 0$, let $v = v(L)$ and $K = K(L)$ be defined as in (3.17), with K increasing in L. As before, let u be a constant multiple of v such that $u \in \mathbb{Z}^d$ and let $(u, u_2, \ldots, u_d)$ be an orthogonal basis for $\mathbb{R}^d$ such that $u_i \in \mathbb{Z}^d$ for all i, and define N as before as well.

We define the graph $\mathcal{H}_{N,L}$ as described before, using the rational direction v, rather than the direction ℓ, to define it. Thus,

$$C_{N,L} := \{x \in \mathbb{Z}^d : 0 \leq x \cdot v \leq L\}/(N\mathbb{Z}u_2, \ldots, N\mathbb{Z}u_d),$$

We note here our reason for using L as the length of the cylinder, rather than $L|u|$ as in [22]. The choice of v depends on the length of the cylinder, but $|u|$ depends on v, and may be unbounded as $v \to \ell$.

As before, arguments based on the graph $\mathcal{H}_{N,L}$ give us

$$\frac{1}{2} \leq \frac{1}{2}\mathbb{P}^M_{\mathcal{H}_{N,L}}(\tilde{T}_\partial = \tau) + \mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 \neq M, T_M < \tilde{T}_\partial), \tag{3.19}$$

and we must show that the term $\mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 \neq M, T_M < \tilde{T}_\partial)$ approaches 0 as L and K increase. Defining $B = B(L, K)$ as before, our previous arguments give us

$$\mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 \neq M, T_M < \tilde{T}_\partial) = \mathbb{P}^0_{\mathcal{G}}(T_{B^c} < T^v_{\leq -R}) \tag{3.20}$$

Comparing with (3.15), the only difference is that the right hand side considers the event $\{T_{B^c} < T^v_{\leq -R}\}$, rather than $\{T_{B^c} < T^\ell_{\leq -R}\}$.

We now must show that (3.20) goes to 0 as L increases (along with N, and with u approaching ℓ). Let $\varepsilon > 0$ and choose $R' > R$. Just as $\mathbb{P}^0_{\mathcal{G}}(T^\ell_{\leq -R} < \infty) = 1$, we have $\mathbb{P}^0_{\mathcal{G}}(T^\ell_{\leq -R'} < \infty) = 1$. Choose $Q = Q(\varepsilon)$ so that $\mathbb{P}^0_{\mathcal{G}}(T_{Q^c} \leq T^\ell_{\leq -R'}) < \varepsilon$. For large enough L, as in (3.16), we have

$$\{T_{B^c} \leq T^v_{\leq -R}\} \subset \{T_{Q^c} \leq T^v_{\leq -R}\}. \tag{3.21}$$

Now by Lemma 3.3.2, for v close enough to ℓ (i.e., for large enough L), if $x \cdot \ell \leq -R'$ and $x \cdot v \geq -R$, then x is not in Q, so that the event $\{T^\ell_{\leq -R'} \leq T_{Q^c} \leq T^v_{\leq -R}\}$ is impossible, and therefore

$$\{T_{Q^c} \leq T^v_{\leq -R}\} \subset \{T_{Q^c} \leq T^\ell_{\leq -R'}\}. \tag{3.22}$$

It follows from (3.21), (3.22), and the choice of Q that for large enough L,

$$\mathbb{P}^0_{\mathcal{G}}(T_{B^c} \leq T^v_{\leq -R}) \leq \mathbb{P}^0_{\mathcal{G}}(T_{Q^c} \leq T^\ell_{\leq -R'}) < \varepsilon.$$

Since this can be true for arbitrary epsilon, $\mathbb{P}^0_{\mathcal{G}}(T_{B^c} \leq T^v_{\leq -R})$ goes to 0, and therefore so does $\mathbb{P}^\partial_{\mathcal{H}_{N,L}}(X_1 \neq M, T_M < \tilde{T}_\partial)$.

Next, we will must show that $\mathbb{P}^M_{\mathcal{H}_{N,L}}(T_\partial = \tau)$ is bounded away from 1 as M increases. Using (3.17) in place of (3.13), we are able to argue as before to get

$$\mathbb{P}^M_{\mathcal{H}_{N,L}}(T_\partial \neq \tau) \geq \frac{1}{2}\alpha.$$

Now taking the limsup in (3.19) as $M \to \infty$ yields the contradiction

$$\frac{1}{2} \leq \frac{1}{2}\left(1 - \frac{1}{2}\alpha\right) < \frac{1}{2}.$$

□

We now have enough to prove Theorem 2.2.1, which we recall here.

Theorem (Theorem 2.2.1)**.** *Let $\mathbb{P}^0_{\mathcal{G}}$ be the measure of a RWDE with bounded jumps on $\mathbb{Z}^d$. Let $\Delta = \mathbb{E}^0[X_1]$ be the annealed drift, and let $\ell \in S^{d-1}$. Then $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 1$ if and only if $\ell \cdot \Delta > 0$; otherwise, $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 0$.*

Proof of Theorem 2.2.1. First, suppose $\Delta \neq 0$. Then if $\ell \cdot \Delta > 0$, the arguments in [22], which work for bounded jumps, show that $\mathbb{P}^0(A_\ell) > 0$. For $d \geq 3$, the proof of the 0-1 law in [14] can easily be modified to work for bounded jumps, as a remark in [22] points out. If $d = 1$, the 0-1 law of [16] applies, and if $d = 2$, Theorem 2.1.1 applies. Thus, we get $\mathbb{P}^0(A_\ell) = 1$. If $\ell \cdot \Delta < 0$, then $-\ell \cdot \Delta > 0$, so we get $\mathbb{P}^0(A_{-\ell}) = 1$, and therefore $\mathbb{P}^0(A_\ell) = 0$. Finally, if $\ell \cdot \Delta = 0$, then the results of [12] (which can easily be made to work for bounded jumps, as noted in the aforementioned remark in [22]) imply that $\mathbb{P}^0(A_\ell) = 0$. This handles the case $\Delta \neq 0$. On the other hand, if $\Delta = 0$, then the conclusion is that of Theorem 2.2.2. □

3.4 Further remarks

We have generalized to RWDE with bounded jumps the complete characterization of $\mathbb{P}^0_{\mathcal{G}}(A_\ell)$ that was known for nearest-neighbor RWDE.

However, there is one nagging difficulty in the zero-drift case that must be dealt with before we may claim absolute victory over the issue of directional transience for RWDE. Because there are uncountably many directions, proving that the probability of transience in any given direction is zero does not automatically mean that it is impossible for the walk to be directionally transient. For example, however unlikely it seems, one could imagine the possibility that a walk is almost surely transient in some direction $\ell \in S^{d-1}$ and recurrent in all directions $\ell' \neq \pm\ell$, but that the direction ℓ of transience is random with a continuous distribution, so that for any fixed ℓ, $\mathbb{P}^0_{\mathcal{G}}(A_\ell) = 0$. This pathological behavior has yet to be ruled out, even for the nearest-neighbor Dirichlet case. To resolve this difficulty we would need to prove, at least for Dirichlet environments, a stronger version of Conjecture 2.2.3. For $\ell \in S^{d-1}$, let A^0_ℓ be the event that $\lim_{n\to\infty} X_n \cdot \ell = \infty$, but there is no neighborhood $U \in S^{d-1}$ containing ℓ such that for all $\ell' \in U$, $\lim_{n\to\infty} X_n \cdot \ell' = \infty$. Using this notation, we can restate Conjecture 2.2.3 (which we have proven in the Dirichlet bounded jump case) very simply.

Conjecture (Conjecture 2.2.3)**.** *Let $\mathbb{P}^0$ be the law of an i.i.d. RWRE on $\mathbb{Z}^d$. Then for all $\ell \in S^{d-1}$, $\mathbb{P}^0(A^0_\ell) = 0$.*

The following strengthened version of the conjecture would rule out the pathological behavior we have described above.

Conjecture 3.4.1. *Let $\mathbb{P}^0$ be the law of an i.i.d. RWRE on $\mathbb{Z}^d$. Then $\mathbb{P}^0\left(\bigcup_{\ell\in S^{d-1}} A^0_\ell\right) = 0$.*

4. BALLISTICITY IN ONE DIMENSION

In this chapter, we characterize ballisticity of RWDE on $\mathbb{Z}$ in terms of the Dirichlet parameters. First we prove Lemma 2.3.2, which gives an abstract characterization of ballisticity for all directionally transient i.i.d. RWRE on $\mathbb{Z}$ with bounded jumps. We then study the parameter κ_0, which characterizes finite traps, and complete the proof of Theorem 2.3.3, which applies Tournier's lemma to our model. We prove Theorem 2.3.4 in two parts, each of which is stated as its own proposition. Finally, we prove Theorem 2.3.5, characterizing finiteness of moments of the quenched Green function $E_\omega^0[N_0]$, and combine it with Lemma 2.3.2 to prove Theorem 2.3.6.

Some of our proofs will require focusing on the environment on a proper subset of the entire state space. For a subset $S \subseteq \mathbb{Z}$, let $\omega^S = (\omega^x)_{x\in S}$. In the case where S is a half-infinite interval, we simplify our notation by using $\omega^{\leq x}$ to denote $\omega^{(-\infty,x]}$, and similarly with $\omega^{<x}$, $\omega^{\geq x}$, and $\omega^{>x}$.

4.1 Abstract ballisticity criteria

The main goal of this section is to prove Lemma 2.3.2, which we recall here.

Lemma (Lemma 2.3.2)**.** *Let P be a probability measure on $\Omega_\mathbb{Z}$ satisfying (C1), (C2), (C3), and (C4). Then $v > 0$ if and only if $\mathbb{E}^0[N_0] = E[E_\omega^0[N_0]] < \infty$.*

Before we can characterize when the almost-sure limiting velocity v is positive, we must first note that it exists. This has been shown under an ellipticity assumption too strong for our model [18], but it can be proven in the more general case with standard techniques. The proof for the recurrent case (where, necessarily, $v = 0$) can be done by a slight modification of arguments in [34]. The proof for the directionally transient case follows [35] in defining *regeneration times* $(\tau_k)_{k=0}^\infty$. Let $\tau_0 := 0$, and for $k \geq 1$, define

$$\tau_k := \min\{n > \tau_{k-1} : X_n > X_j \text{ for all } j < n, X_n \leq X_j \text{ for all } j > n\}. \tag{4.1}$$

A crucial fact is that the sequences $(X_{\tau_n} - X_{\tau_{n-1}})_{n=2}^{\infty}$ and $(\tau_n - \tau_{n-1})_{n=2}^{\infty}$ are i.i.d. Using these regeneration times, we are able to derive a formula for v, as well as a characterization in terms of hitting times.

Proposition 4.1.1. *Let P be a probability measure on $\Omega_{\mathbb{Z}}$ satisfying (C1), (C2), and (C3). Then the following hold:*

1. *There is a $\mathbb{P}^0$–almost sure limiting velocity*

$$v := \lim_{n\to\infty} \frac{X_n}{n} = \frac{\mathbb{E}^0[X_{\tau_2} - X_{\tau_1}]}{\mathbb{E}[\tau_2 - \tau_1]}, \tag{4.2}$$

   where the numerator is always finite, and the fraction is understood to be 0 if the denominator is infinite.

2. $\lim_{x\to\infty} \frac{T_{\geq x}}{x} = \frac{1}{v}$, *where $\frac{1}{v}$ is understood to be ∞ if $v = 0$.*

We outline some details of the argument for Proposition 4.1.1 in Chapter 5. However, the argument there is designed to apply also to accelerated, continuous-time random walks as well as the discrete-time case. For that reason, we outline a somewhat simplified version in Appendix A that applies only to the discrete-time case.

For the rest of this section, assume P satisfies (C1), (C2), (C3), and (C4). We also use regeneration times to derive the following lemma.

Lemma 4.1.2. *For any $a, c \in \mathbb{Z}$,*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{k=c}^{x} N_k = \frac{1}{v}, \ \mathbb{P}^a\text{–a.s.}$$

If $v = 0$, then the limit is infinity.

Proof. Fix a. Recall that $N_k^{(-\infty,x)}$ is the amount of time the walk spends at k before $T_{\geq x}$. Then for $x > c$,

$$\frac{T_{\geq x}}{x} = \frac{1}{x} \sum_{k=-\infty}^{c-1} N_k^{(-\infty,x)} + \frac{1}{x} \sum_{k=c}^{x-1} N_k^{(-\infty,x)}.$$

The first term approaches 0 almost surely by assumption (C4); hence, by Proposition 4.1.1 (2),

$$\lim_{x\to\infty} \frac{1}{x} \sum_{k=c}^{x-1} N_k^{(-\infty,x)} = \frac{1}{v}, \mathbb{P}^a\text{–a.s.} \tag{4.3}$$

We note that N_k and $N_k^{(-\infty,x)}$ differ only if the walk backtracks and visits k after reaching $[x, \infty)$. The sum, over all $k < x$, of these differences, is the total amount of time the walk spends to the left of x after $T_{\geq x}$, and it is bounded above by the time from $T_{\geq x}$ to the next regeneration time (defined as in (4.1)), which is in turn bounded above by $\tau_{J(x)} - \tau_{J(x)-1}$, where $J(x)$ is the (random) j such that $\tau_{j-1} \leq T_{\geq x} < \tau_j$. Hence

$$\frac{1}{x} \sum_{k=c}^{x-1} N_k^{(-\infty,x)} \leq \frac{1}{x} \sum_{k=c}^{x-1} N_k \leq \frac{1}{x} \sum_{k=c}^{x-1} N_k^{(-\infty,x)} + \frac{1}{x}[\tau_{J(x)} - \tau_{J(x)-1}] \tag{4.4}$$

Assume $v = 0$. Then by (4.3), the left side of (4.4) approaches ∞ as x approaches ∞, and therefore so does the middle. On the other hand, suppose $v > 0$. By (4.2), $\mathbb{E}[\tau_2 - \tau_1] < \infty$. Then by the strong law of large numbers, $\frac{\tau_n}{n} \to \mathbb{E}[\tau_2 - \tau_1] < \infty$, which implies that $\frac{\tau_n - \tau_{n-1}}{n}$ approaches 0. Since $J(x) \leq x + 1$, the term $\frac{1}{x}[\tau_{J(x)} - \tau_{J(x)-1}]$ approaches zero almost surely; hence the Squeeze Theorem yields the desired result. $\square$

Suppose for now that $R = 1$. Then, for almost every ω, it is possible to define a bi-infinite walk $\overline{\mathbf{X}} = (\overline{X}_n)_{n\in\mathbb{Z}}$ whose "right halves" are distributed like random walks under ω. From each site a, run a walk according to the transition probabilities given by ω until it reaches $a + 1$ (which occurs in finite time P_ω^a–a.s. for P–a.e. ω). Concatenating all of these walks then gives, up to a time shift[1], a unique walk $\overline{\mathbf{X}} = (\overline{X}_n)_{n\in\mathbb{Z}}$ such that for any $x \in \mathbb{Z}$, the distribution of $(\overline{X}_k)_{k=n}^{\infty}$, conditioned on $\overline{X}_n = x$, is P_ω^x. We may think of $\overline{\mathbf{X}}$ as a walk from $-\infty$ to ∞ in the environment ω.

With a bit more work, we can define a similar bi-infinite walk in the general case $R > 0$. Call the set of vertices $((k-1)R, kR]$ the kth *level* of $\mathbb{Z}$, and for $x \in \mathbb{Z}$, let $[[x]]_R$ denote the level containing x. Let ω be a given environment. From each point $a \in \mathbb{Z}$, run a walk according to the transition probabilities given by ω until it reaches the next level (i.e.,

[1]↑Choose, for example, the time shift where $\overline{X}_0 = 0$ and where $\overline{X}_n < 0$ whenever $n < 0$.

$[[a+R]]_R$). This will happen P_ω^a–a.s. for P–a.e. ω, by transience to the right and because it is not possible to jump over a set of length R. Do this independently at every point for every level. This gives what we'll call a *cascade*: a set of (almost surely finite) walks indexed by $\mathbb{Z}$, where the walk indexed by $a \in \mathbb{Z}$ starts at a and ends upon reaching level $[[a+R]]_R$. Then for almost every cascade, concatenating these finite walks gives, for each point a, a right-infinite walk $\mathbf{X}^a = (X_n^a)_{n=0}^\infty$. Let P_ω be the probability measure we have just described on the space of cascades, and let $\mathbb{P} = P \times P_\omega$.

It is crucial to note that by the strong Markov property, the law of $\mathbf{X}^a$ under P_ω is the same as the law of $\mathbf{X}$ under P_ω^a, which also implies that the law of $\mathbf{X}^a$ under $\mathbb{P}$ is the same as the law of $\mathbf{X}$ under $\mathbb{P}^a$.

For each $x \in \mathbb{Z}$, let the "coalescence event" C_x be the event that all the walks from level $[[x-R]]_R$ first hit level $[[x]]_R$ at x. On the event C_x, we say a coalescence occurs at x.

Lemma 4.1.3. *Let $\mathcal{E}_1$ be the event that all the $\mathbf{X}^a$ are transient to the right, that all steps to the left and right are bounded by L and R, respectively, and that infinitely many coalescences occur to the left and to the right of 0. Then $\mathbb{P}(\mathcal{E}_1) = 1$.*

Proof. Boundedness of steps has probability 1 by assumption (C3), and by assumption (C4) all the walks $\mathbf{X}^a$ are transient to the right with probability 1. Now for $k \geq 2$ and $x \in \mathbb{Z}$, let $C_{x,k}$ be the event that all the walks from level $[[x-R]]_R$ first hit level $[[x]]_R$ at x without ever having reached $[[x-kR]]_R$. Choose k large enough that $\mathbb{P}(C_{0,k}) > 0$; then under the law $\mathbb{P}$, the events $\{C_{nkR,k}\}_{n\in\mathbb{Z}}$ are all independent and have equal, positive probability. Thus, infinitely many of them will occur in both directions, $\mathbb{P}$–a.s. By definition, $C_{x,k} \subset C_x$, and so infinitely many of the events C_x occur in both directions, $\mathbb{P}$–a.s. $\square$

Assume the environment and cascade are in the event $\mathcal{E}_1$. Let $(x_k)_{k\in\mathbb{Z}}$ be the locations of coalescence events (with x_0 the smallest non-negative x such that C_x occurs). By definition of the x_k, for every k and for every a to the left of $[[x_k]]_R$, $T_{[[x_k]]_R}(\mathbf{X}^a) = T_{x_k}(\mathbf{X}^a) < \infty$. Now for $j < k$, it necessarily holds that x_j is to the left of $[[x_k]]_R$, since there can be only one x_k per level. Define $\nu(j,k) := T_{x_k}(\mathbf{X}^{x_j})$. By definition of the walks $\mathbf{X}^a$, we have for $j < k$, $n \geq 0$,

$$X_{n+\nu(j,k)}^{x_j} = X_n^{x_k}. \tag{4.5}$$

From this one can easily check that the $\nu(j,k)$ are additive; that is, for $j<k<\ell$, we have $\nu(j,\ell)=\nu(j,k)+\nu(k,\ell)$.

Because all the $\mathbf{X}^{x_k}$ agree with each other in the sense of (4.5), we may define a single, bi-infinite walk $\overline{\mathbf{X}}=(\overline{X}_n)_{n\in\mathbb{Z}}$ that agrees with all of the $\mathbf{X}^{x_k}$. For $n\geq 0$, let $\overline{X}_n=X_n^{x_0}$. For $n<0$, choose $j<0$ such that $\nu(j,0)>|n|$, and let $X_n=X^{x_j}_{\nu(j,0)-|n|}$. This definition is independent of the choice of j, because if $j<k<0$ with $v(k,0)>|n|$, then by (4.5) and the additivity of the $\nu(j,k)$, we have

$$X^{x_j}_{\nu(j,0)-|n|}=X^{x_j}_{\nu(j,k)+\nu(k,0)-|n|}=X^{x_k}_{\nu(k,0)-|n|}.$$

We may then define $\overline{N}_x:=\#\{n\in\mathbb{Z}:X_n=x\}$ to be the amount of time the walk $\overline{\mathbf{X}}$ spends at x. Thus, $\overline{N}_x=\lim_{a\to-\infty}N_x(\mathbf{X}^a)$.

Lemma 4.1.4. *Both of the sequences $(\mathbf{X}^a)_{a\in\mathbb{Z}}$ and $(\overline{N}_x)_{x\in\mathbb{Z}}$ are stationary and ergodic.*

Proof. For a given environment, the cascade that defines $\overline{\mathbf{X}}$ may be generated by a (countable) family $\mathbf{U}=(U_n^a)_{n\in\mathbb{N},a\in\mathbb{Z}}$ of i.i.d. uniform random variables on $[0,1]$. For such a collection, and an $a\in\mathbb{Z}$, let $\mathbf{U}^a$ be the projection $(U_n^a)_{n\in\mathbb{N}}$. Given an environment ω, the finite walk from a to level $[[a+R]]_R$ may be generated using the first several U_n^a. (One of the U_n^a is used for each step. Once the walk terminates, the rest of the U_n^a are not needed, but one does not know in advance how many will be needed.) Let $\hat{\omega}^x=(\omega^x,\mathbf{U}^x)$, and $\hat{\omega}=(\hat{\omega}^x)_{x\in\mathbb{Z}}$. Define the left shift $\hat{\theta}$ by $\hat{\theta}(\hat{\omega}):=(\hat{\omega}^{x+1})_{x\in\mathbb{Z}}$. Then $(\hat{\omega}^x)_{x\in\mathbb{Z}}$ is an i.i.d. sequence. We have $\mathbf{X}^0=\mathbf{X}^0(\hat{\omega})$ and $\mathbf{X}^a=\mathbf{X}^0(\hat{\theta}^a\hat{\omega})$. Similarly, $\overline{N}_0=\overline{N}_0(\hat{\omega})$ and $N_x=\overline{N}_0(\hat{\theta}^x\hat{\omega})$. So it suffices to show that $\mathbf{X}^0$ and $\overline{N}_0$ are measurable. The measurability of $\mathbf{X}^0$ is obvious. For $\overline{N}_0$, let $A_{k,\ell,B,r}$ be the event that:

(a) for some $x<0$, a coalescence event $C_{x,k}$ (as defined in the proof of Lemma 4.1.3) occurs with $-B\leq x-kR<x<0$, so that $\overline{\mathbf{X}}$ agrees with $\mathbf{X}^x$ to the right of x;

(b) $N_0^{[-B,B]}(\mathbf{X}^x)\geq\ell$, where $N_0^{[-B,B]}$ is the amount of time the walk spends at x before exiting $[-B,B]$; and

(c) none of the walks from sites $a\in[-B,B]$ uses more than r of the random variables U_r^a.

On this event, $\overline{N}_0$ is seen to be at least ℓ by looking only within $[-B, B]$ and only at the first r uniform random variables at each site. The event $A_{k,\ell,B,r}$ is measurable, because it is a measurable function of finitely many random variables, and the event $\{\overline{N}_0 > \ell\}$ is, up to a null set, simply the union over all r, then over all B, and then over all k of these events. Thus, $\overline{N}_0$ is measurable. $\square$

We now give the connection between $\overline{N}_0$ and the limiting velocity v.

Lemma 4.1.5. $v = \frac{1}{\mathbb{E}[\overline{N}_0]}$. *Consequently, the walk is ballistic if and only if* $\mathbb{E}[\overline{N}_0] < \infty$.

We note that a similar formula for the limiting speed in the ballistic case can be obtained from [36, Theorem 6.12] for discrete-time RWRE on a strip, although the probabilistic interpretation is less explicit, and an ellipticity assumption that does not hold for Dirichlet RWRE is required.

Proof. By Lemma 4.1.4 and Birkhoff's Ergodic theorem, for any $c \in \mathbb{Z}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=c}^{n} \overline{N}_k = \mathbb{E}[\overline{N}_0],\ \mathbb{P}\text{–a.s.}$$

Fix $a \in \mathbb{Z}$. For large enough k, $N_k(\mathbf{X}^a) = \overline{N}_k$. We therefore get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=c}^{n} N_k(\mathbf{X}^a) = \mathbb{E}[\overline{N}_0],\ \mathbb{P}\text{–a.s.}$$

It follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=c}^{n} N_k(\mathbf{X}) = \mathbb{E}[\overline{N}_0],\ \mathbb{P}^a\text{–a.s.}$$

By Lemma 4.1.2, we get $v = \frac{1}{\mathbb{E}[\overline{N}_0]}$. $\square$

Now we can see that the walk is ballistic if and only if $\mathbb{E}[\overline{N}_0] < \infty$. In order to prove Lemma 2.3.2, we need to compare $\mathbb{E}[\overline{N}_0]$ with $\mathbb{E}^0[N_0]$.

Lemma 4.1.6. $\mathbb{E}[\overline{N}_0] \leq \mathbb{E}^0[N_0]$.

Proof. If $\mathbb{E}^0[N_0] = \infty$, the inequality is trivial. Assume, therefore, that $\mathbb{E}^0[N_0] < \infty$.

Note that $\lim_{x\to\infty} N_0(\mathbf{X}^{-x}) = \overline{N}_0$, $\mathbb{P}$–a.s. Assuming we are able to interchange a limit with an expectation, we have

$$\begin{aligned}
\mathbb{E}[\overline{N}_0] &= \mathbb{E}\left[\lim_{x\to\infty} N_0(\mathbf{X}^{-x})\right] \\
&= \lim_{x\to\infty} \mathbb{E}\left[N_0(\mathbf{X}^{-x})\right] \\
&= \lim_{x\to\infty} \mathbb{E}^{-x}[N_0(\mathbf{X})].
\end{aligned} \tag{4.6}$$

But each term $\mathbb{E}^{-x}[N_0(\mathbf{X})] = E[E_\omega^{-x}[N_0]]$ is less than $\mathbb{E}^0[N_0] = E[E_\omega^0[N_0]]$, since $E_\omega^{-x}[N_0] = P_\omega^{-x}(T_0 < \infty)E_\omega^0[N_0]$. Therefore, we may conclude $\mathbb{E}[\overline{N}_0] \leq \mathbb{E}^0[N_0]$, provided we can justify (4.6). To do this, we will apply the dominated convergence theorem, noting that $N_0(\mathbf{X}^{-x}) \leq \max_{1-R\leq y\leq 0} N_0(\mathbf{X}^y)$ for all $x > R$. To see that the latter has finite expectation, we have

$$\begin{aligned}
\mathbb{E}\left[\max_{1-R\leq y\leq 0} N_0(\mathbf{X}^y)\right] &\leq \sum_{y=1-R}^{0} \mathbb{E}\left[N_0(\mathbf{X}^y)\right] \\
&= \sum_{y=1-R}^{0} E\left[E_\omega[N_0(\mathbf{X}^y)]\right] \\
&= \sum_{y=1-R}^{0} E\left[E_\omega^y[N_0]\right] \\
&= \sum_{y=1-R}^{0} E\left[P_\omega^y(T_0 < \infty)E_\omega^0[N_0]\right] \\
&\leq \sum_{y=1-R}^{0} E\left[E_\omega^0[N_0]\right] \\
&= R\mathbb{E}^0[N_0] \\
&< \infty.
\end{aligned}$$

This allows us to justify our use of the dominated convergence theorem, completing the proof. $\square$

We must now handle the case where $\mathbb{E}^0[N_0] = \infty$. Our first step is to prove Lemma 2.3.1.

Lemma (Lemma 2.3.1)**.** *Let P be a probability measure on $\Omega_{\mathbb{Z}}$ satisfying (C1), (C2), (C3), and (C4). Then $v > 0$ if and only if $\mathbb{E}^0[T_{\geq 1}] < \infty$, where $T_{\geq 1}$ is the first time the walk hits $[1, \infty)$.*

Proof of Lemma 2.3.1. Suppose $\mathbb{E}^0[T_{\geq 1}(\mathbf{X})] < \infty$. Then for P_ω–almost every cascade, we have

$$\begin{aligned}\frac{T_{\geq x}(\mathbf{X}^0)}{x} &= \frac{1}{x}\sum_{k=1}^{x}(T_{\geq k}(\mathbf{X}^0) - T_{\geq k-1}(\mathbf{X}^0)) \\ &\leq \frac{1}{x}\sum_{k=1}^{x} T_{\geq k}(\mathbf{X}^{k-1}), \end{aligned} \tag{2.7}$$

where the inequality comes from the fact that if $\mathbf{X}^0$ hits $[k-1, \infty)$ at $k-1$, then it follows the same path from there as $\mathbf{X}^{k-1}$, while if it hits $[k-1, \infty)$ at a point to the right of $k-1$, then $T_{\geq k} - T_{\geq k-1} = 0$. By Birkhoff's Ergodic Theorem, the right side $\mathbb{P}$–a.s. approaches $\mathbb{E}[T_{\geq 1}(\mathbf{X}^0)] = \mathbb{E}^0[T_{\geq 1}]$. Now we know from Proposition 4.1.1 that $\lim_{n\to\infty} \frac{n}{X_n} = \frac{1}{v}$, so the subsequence $\frac{T_{\geq x}(\mathbf{X}^0)}{X^0_{T_{\geq x}}}$ must have the same limit. Since, for $x > R$, we have

$$\frac{T_{\geq x}(\mathbf{X}^0)}{X^0_{T_{\geq x}} + R} \leq \frac{T_{\geq x}(\mathbf{X}^0)}{x} \leq \frac{T_{\geq x}(\mathbf{X}^0)}{X^0_{T_{\geq x}} - R},$$

we get $\frac{1}{v} = \lim_{x\to\infty} \frac{T_{\geq x}(\mathbf{X}^0)}{x}$. Applying (2.7), we get

$$\begin{aligned}\frac{1}{v} &= \lim_{n\to\infty} \frac{T_{\geq x}(\mathbf{X}^0)}{x} \\ &\leq \lim_{n\to\infty} \frac{1}{x}\sum_{k=1}^{x} T_{\geq k}(\mathbf{X}^{k-1}) \\ &= \mathbb{E}^0[T_{\geq 1}].\end{aligned}$$

Therefore, if $\mathbb{E}^0[T_{\geq 1}] < \infty$, then $v > 0$.

On the other hand, suppose $\mathbb{E}^0[T_{\geq 1}] = \infty$. We will show that $v = 0$.

Claim 2.3.1.1. $\mathbb{E}\left[\min_{1\leq i\leq R} T_{\geq R+1}(\mathbf{X}^i)\right] = \infty$.

By assumptions (C1), (C2), and (C3), we still have an $m_0 \geq \max(L, R)$ large enough that every interval of length m_0 is irreducible, P–a.s. Let A be the event that

- For each $i = 1, \ldots, R-1$, the walk $\mathbf{X}^i$ hits R before leaving $[R - m_0 + 1, R]$.

- The walk $\mathbf{X}^R$ first exits $[R - m_0 + 1, R]$ by hitting $R - m_0$.

Then under P, the quenched probability of A is independent of $\omega^{\leq R - m_0}$. Now, on the event A, the minimum $\min_{1 \leq i \leq R} T_{\geq R+1}(\mathbf{X}^i)$ is attained for $i = R$, since all the other walks take time to get to R and then simply follow $\mathbf{X}^R$. Now on A, $T_{\geq R+1}(\mathbf{X}^R)$ is greater than the amount of time it takes for the walk $\mathbf{X}^R$ to cross back to $[R - m_0 + 1, \infty)$ after first hitting $R - m_0$. The quenched expectation of this time, conditioned on A, is $E_\omega^{R-m_0}[T_{\geq R - m_0 + 1}]$ by the strong Markov property, and this depends only on $\omega^{\leq R - m_0}$. Hence

$$
\begin{aligned}
\mathbb{E}\left[\min_{1 \leq i \leq R} T_{\geq R+1}(\mathbf{X}^i)\right] &\geq E\left[P_\omega(A) E_\omega[T_{\geq R+1}(\mathbf{X}^R)|A]\right] \\
&\geq E\left[P_\omega(A) E_\omega^{R-m_0}[T_{\geq R-m_0+1}(\mathbf{X})]\right] \\
&= \mathbb{P}(A)\mathbb{E}^{R-m_0}[T_{\geq R-m_0+1}(\mathbf{X})] \\
&= \mathbb{P}(A)\mathbb{E}^0[T_{\geq 1}(\mathbf{X})] \\
&= \infty.
\end{aligned}
$$

This proves our claim. Now for $x \geq 1$,

$$
T_{\geq xR+1}(\mathbf{X}^0) \geq T_{\geq 1}(\mathbf{X}^0) + \sum_{k=1}^{x} \min_{1 \leq i \leq R} T_{\geq kR+1}(\mathbf{X}^{(k-1)R+i}).
$$

Dividing by xR and taking limits as $x \to \infty$, we get $\lim_{x\to\infty} \frac{T_{\geq xR+1}(\mathbf{X}^0)}{xR} = \infty$, $\mathbb{P}$–a.s. by Birkhoff's ergodic theorem. Hence $\lim_{x\to\infty} \frac{T_{\geq xR+1}(\mathbf{X})}{xR+1} = \infty$, $\mathbb{P}^0$–a.s. It follows that $v = 0$. □

Now we can handle the case $\mathbb{E}^0[N_0] = \infty$.

Proposition 4.1.7. *If* $\mathbb{E}^0[N_0] = \infty$*, then* $v = 0$.

Proof. Suppose $\mathbb{E}^0[N_0] = \infty$. We want to show that $v = 0$. By Lemma 2.3.1, it suffices to show that $\mathbb{E}^0[T_{\geq 1}] = \infty$.

Now N_0 is the total number of visits the walk makes to 0. These visits may be sorted based on the farthest point to the right that the walk has hit in the past at the time of each visit. As in the proof of Lemma 4.1.2, we use $N_k^{(-\infty,x)}$ to denote the amount of time the walk spends at k before $T_{\geq x}$. Thus, for a walk started at 0 we get

$$N_0 = \sum_{x=0}^{\infty} \left(N_0^{(-\infty,x+1)} - N_0^{(-\infty,x)} \right). \tag{4.8}$$

Taking expectations on both sides, we get

$$\mathbb{E}^0[N_0] = \sum_{x=0}^{\infty} E\left[E_\omega^0 \left[N_0^{(-\infty,x+1)} - N_0^{(-\infty,x)} \right] \right] \tag{4.9}$$

Now $N_0^{(-\infty,x)}$ and $N_0^{(-\infty,x+1)}$ can only differ if the walk hits $[x,\infty)$ at x. Conditioned on this event, the distribution under P_ω^0 of the walk $(X_{n+T_{\geq x}})_{n=0}^{\infty}$ is the distribution of $\mathbf{X}$ under P_ω^x. Thus,

$$E_\omega^0 \left[N_0^{(-\infty,x+1)} - N_0^{(-\infty,x)} \right] = P_\omega^0(X_{T_{\geq x}} = x) E_\omega^x \left[N_0^{(-\infty,x+1)} \right]. \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$\begin{aligned} \mathbb{E}^0[N_0] &= \sum_{x=0}^{\infty} E\left[P_\omega^0(X_{T_{\geq x}} = x) E_\omega^x \left[N_0^{(-\infty,x+1)} \right] \right] \\ &\leq \sum_{x=0}^{\infty} E\left[E_\omega^x \left[N_0^{(-\infty,x+1)} \right] \right] \\ &= \sum_{x=0}^{\infty} \mathbb{E}^x \left[N_0^{(-\infty,x+1)} \right]. \end{aligned}$$

By stationarity,

$$\begin{aligned} \mathbb{E}^0[N_0] &\leq \sum_{x=0}^{\infty} \mathbb{E}^0 \left[N_{-x}^{(-\infty,1)} \right] \\ &= \mathbb{E}^0 \left[\sum_{x=0}^{\infty} N_{-x}^{(-\infty,1)} \right] \\ &= \mathbb{E}^0[T_{\geq 1}]. \end{aligned}$$

If $\mathbb{E}^0[N_0] = \infty$, it follows that $\mathbb{E}^0[T_{\geq 1}]$, and by Lemma 2.3.1, $v = 0$. $\square$

We can now complete the proof of our main lemma.

Proof of Lemma 2.3.2. Assume the walk is transient to the right. If $\mathbb{E}^0[N_0] = \infty$, the conclusion is that of Proposition 4.1.7. Otherwise, combining Lemmas 4.1.5 and 4.1.6 gives $v > 0$. The left-transient case follows by symmetry. By the 0-1 law of [16], the remaining case is where the walk is recurrent. This implies that $N_0 = \infty$, $\mathbb{P}^0$–a.s., so that $\mathbb{E}^0[N_0] = \infty$. Because $v = 0$ in the recurrent case, the lemma is true. $\square$

4.2 Parameters governing ballisticity of RWDE

We now return to the Dirichlet model. In this section, we will characterize ballisticity in terms of L, R, and the parameters $(\alpha_i)_{i=-L}^{R}$. Lemma 2.3.2 tells us that the walk is ballistic precisely when the quantity $\mathbb{E}^0_{\mathcal{G}}[N_0] = E_{\mathcal{G}}[E^0_\omega[N_0]]$ is finite. Although we cannot usually calculate this expectation, we are able to characterize when it is finite in terms of our Dirichlet parameters. We assume throughout this section that $\kappa_1 > 0$, so that the walk is transient to the right, and we examine the integrability of $E^0_\omega[N_0]$ under $P_{\mathcal{G}}$. In fact, we generalize the question, examining when $E_{\mathcal{G}}[E^0_\omega[N_0]^s] < \infty$ for $s > 0$. The goal of this section is to prove Theorems 2.3.3, 2.3.4, and 2.3.5. Theorem 2.3.6 then easily follows from Theorem 2.3.5 and Lemma 2.3.2.

4.2.1 Finite traps: The parameter κ_0

In this subsection, we use Tournier's lemma to study the existence of finite traps—finite sets in which the walk is expected to spend an infinite amount of time before exiting under the annealed measure.

We note that although Tournier's lemma is stated for a finite graph, it can readily be applied to walks that are killed upon exiting a finite subset of an infinite graph. In particular, for any positive integer M, we can identify all vertices outside of $[-M, M]$ into one sink and then argue as in Claim 3.2.1.1 or use Lemma 1.4.1 to see that the annealed expected number of visits to 0 before exiting $[-M, M]$ is infinite if and only if there is a strongly connected subset S of $[-M, M]$ containing 0 such that $\beta_S \leq 1$.

This discussion motivates us to define, for our graph $\mathcal{G}$, the quantity

$$\kappa_0 := \inf\{\beta_S : S \subset \mathbb{Z} \text{ finite, strongly connected}\}. \tag{4.11}$$

We now recall Theorem 2.3.3. (Also recall that N_0^S is the amount of time the walk spends at 0 before first exiting S.)

Theorem (Theorem 2.3.3)**.** *For $s > 0$, the following are equivalent:*

(a) $\kappa_0 \leq s$*.*

(b) For all sufficiently large M, $E_{\mathcal{G}}\left[E_\omega^0\left[N_0^{[-M,0]}\right]^s\right] = \infty$.

(c) For some $M \geq 0$, $E_{\mathcal{G}}\left[E_\omega^0\left[N_0^{[-M,M]}\right]^s\right] = \infty$.

The proof is essentially a straightforward application of Tournier's lemma as we have just discussed; however, in order to handle the boundary case $s = \kappa_0$, we first need to show that the infimum in the definition of κ_0 is actually a minimum. For example, in the case $\kappa_0 = s = 1$, showing that κ_0 is a minimum means showing that there is actually a finite set containing 0 that the walk is expected to get stuck in for an infinite amount of time. We also give an algorithm to compute κ_0.

Proposition 4.2.1. *The infimum κ_0 for the graph $\mathcal{G}$ is actually a minimum attained by a set $S \subset \mathbb{Z}$. Moreover, there is an integer M, which may be calculated from L, R, and the weight assignments $(\alpha_i)_{-L \leq i \leq R}$, such that the infimum is attained on a subset S with diameter at most M. Hence κ_0 can be calculated directly.*

By translation invariance of the graph $\mathcal{G}$, this implies that it is possible to compute κ_0 by looking at strongly connected subsets S with diameter no more than M and with leftmost point 0.

Proof. We prove this in a series of claims. Recall that $m_0 \geq \max(L, R)$ is large enough that every interval of length m_0 is strongly connected.

Claim 4.2.1.1. $\kappa_0 \leq d^+ + d^-$.

To prove this claim, it suffices to exhibit a finite, strongly connected set $S \subset \mathbb{Z}$ with $\beta_S = d^+ + d^-$. Let $S = [0, m_0 - 1]$. Then S is strongly connected. Now β_S is the total weight of edges from $[0, m_0 - 1]$ to other vertices. Since $m \geq \max(L, R)$, it is easy to check that this is exactly $d^+ + d^-$.

Claim 4.2.1.2. Let $S \subset \mathbb{Z}$ be a finite, strongly connected set of vertices. If x is a vertex to the left or to the right of S, then $\beta_{S \cup \{x\}} \geq \beta_S$.

The quantity β_S is the sum of all weights from vertices in S to vertices not in S. The quantity $\beta_{S \cup \{x\}}$ counts all same weights, except for weights of edges from S to x, and it also counts weights of edges from x to vertices not in $S \cup \{x\}$. If x is to the right of S, then the total weight of edges from S to x cannot be more than c^+, because c^+ is the total weight into x from all vertices to the left of x. On the other hand, c^+ is also the total weight from x to all vertices to the right of x, which are necessarily not in $S \cup \{x\}$. Thus, the additional weight from x to the right at least makes up for any weight into x from S. This proves the claim in the case that x is to the right of A, and a similar argument proves the symmetric case.

Remark 4.2.1. Note the importance of the assumption that x is to the left or right of S. If x is in between some of the vertices of S, then it is certainly possible that $\beta_{S \cup \{x\}} < \beta_S$. See Examples B.0.7 and B.0.8.

Claim 4.2.1.3. Let S be a finite, strongly connected subset of $\mathbb{Z}$. Say that a vertex $x \in S$ is *insulated* if every site reachable in one step from x is also in S. Then if $x < y$ are consecutive non-insulated vertices in S with $y - x > m_0$, it must be the case that all vertices between x and y are in S.

Suppose there are two consecutive non-insulated vertices $x, y \in S$ with $y - x > m_0$. Because $m_0 \geq \max(L, R)$, there must be other vertices from S strictly between x and y in order for S to contain a path from x to y and y to x. By assumption, all such vertices are insulated. Therefore, if there is an edge from a vertex in $(x, y) \cap S$ to another vertex in (x, y), then the latter vertex must also be in S, and since it is strictly between x and y, it must also be insulated. Applying this fact repeatedly, we see that any two vertices that communicate within (x, y) are either both insulated in S or both in S^c. Since the length of (x, y) is at least m_0, all sites in the interval communicate, and so all are in S.

Claim 4.2.1.4. The infimum κ_0 is attained as a minimum; $\kappa_0 = \beta_{S_0}$ for some S_0. Moreover, there is an algorithm to find it.

Let ε be the smallest weight any edge in $\mathcal{G}$ has, let N be an integer such that $N\varepsilon \geq d^+ + d^-$, and let $M = (N-1)(m_0)$ (note this implies $M \geq m_0$). Then if S is a set of vertices with diameter greater than M, it must either have at least N non-insulated vertices or have consecutive non-insulated vertices that differ by more than m_0. If there are at least N non-insulated vertices, then there is an edge from each of these to at least one vertex outside of S, which means $\beta_S \geq N\varepsilon \geq d^+ + d^- \geq \kappa_0$, the last inequality coming from Claim 4.2.1.1. On the other hand, if two consecutive non-insulated vertices $x < y$ differ by more than m_0, then $[x, y] \subseteq S$ by Claim 4.2.1.3. Now $\beta_{[x,y]} = d^+ + d^-$, and vertices to the left and to the right of $[x, y]$ can only increase β_S by Claim 4.2.1.2. Thus, $\beta_S \geq d^+ + d^- \geq \kappa_0$. Therefore, one can compute κ_0 by looking only at β_S for subsets S of $\mathbb{Z}$ with diameter no larger than M (note that this includes $[0, m_0 - 1]$, which has $\beta_{[0,m_0-1]} = d^+ + d^-$). By shift invariance, one can in fact look only at subsets of $[0, M]$. Since there are only finitely many such sets, the infimum in the definition of κ_0 is actually a minimum. Since a suitable M can be easily calculated from L, R, and the α_i, finding such an M and then examining all strongly connected subsets of $\mathbb{Z}$ with leftmost point 0 and diameter $\leq M$ gives us an algorithm to find κ_0. $\square$

We are now able to prove Theorem 2.3.3.

Proof of Theorem 2.3.3.

(a) $\Rightarrow$ (b) Suppose $\kappa_0 \leq s$. By Proposition 4.2.1, this means there is a finite, strongly connected set S of vertices in $\mathbb{Z}$ such that $\beta_S \leq s$. By translation invariance, we may assume 0 is the rightmost point of S. Let $-M$ be large enough that $S \subseteq [-M, 0]$. By collapsing all vertices not in $[-M, 0]$ into a sink and then arguing as in Claim 3.2.1.1, we may apply Tournier's lemma along with the amalgamation property to see that $E_{\mathcal{G}}\left[E_\omega^0\left[N_0^{[-M,0]}\right]^s\right] = \infty$.

(b) $\Rightarrow$ (c) Immediate, since $N_0^{[-M,0]} \leq N_0^{[-M,M]}$.

(c) $\Rightarrow$ (a) Suppose $E_{\mathcal{G}}\left[E_\omega^0\left[N_0^{[-M,M]}\right]^s\right] = \infty$. Again, collapsing all vertices not in $[-M,M]$ into a single sink, we may apply Tournier's lemma to see that there is a strongly connected set $S \subseteq [-M,M]$ such that $\beta_S \leq s$. Hence $\kappa_0 \leq s$. $\square$

The method for finding κ_0 given in the proof of Proposition 4.2.1 requires knowledge of the α_i, and the number of sets to examine grows exponentially with the reciprocal of the smallest positive α_i. We prove in Appendix B that given only the set $\mathcal{N}$ (that is, given only L, R, and the i for which $\alpha_i > 0$), κ_0 can be expressed as a minimum of finitely many positive integer combinations of the α_i. If one has this formula, then one may easily compute κ_0 for any specific values of the α_i.

Proposition 4.2.2. *Given the set $\mathcal{N}$, κ_0 is an elementary function (a minimum of finitely many positive integer combinations) of the α_i.*

Jump to proof.

Notice that Proposition 4.2.2 would have sufficed in place of Proposition 4.2.1 to show that the infimum in the definition of κ_0 is a minimum, which is enough to prove Theorem 2.3.3. On the one hand, Proposition 4.2.2 is stronger than Proposition 4.2.1 in that it shows, given the structure of the graph $\mathcal{G}$, that there is an elementary formula for κ_0 that holds for all possible choices of α_i, whereas Proposition 4.2.1 gives κ_0 as a minimum of β_S for sets S in a collection of sets that is finite, but whose size depends on the sizes of the α_i. On the other hand, Proposition 4.2.2 is weaker than Proposition 4.2.1 in that it does not lead to an explicit algorithm for finding κ_0, since we do not have an algorithm to find the formula that is proven to exist in Proposition 4.2.2. Nevertheless, we give examples in Appendix B where we are able to find this formula. We leave it as an open question to find a general algorithm to do this.

4.2.2 Large-scale backtracking: The parameter κ_1

We have seen that the parameter κ_0 controls moments of the quenched expected time a walk spends at 0 before exiting a finite set. We will now show that in a similar way, κ_1 controls moments of backward traversals of arbitrarily large stretches of the graph.

In discussing the proof of the transient, one-dimensional case of Theorem 2.2.1, we used the graphs $\mathcal{G}_M$, finite graphs that looked like $\mathcal{G}$ except near endpoints. Here, we consider these along with a "limiting graph" that is half infinite. Let $\mathcal{G}_+$ be a graph with vertex set $[0,\infty)$. The graph $\mathcal{G}_+$ contains all the same edges between vertices to the right of 0 with the same weights as $\mathcal{G}$. For vertices $1 \le i \le L$, there is an edge from i to 0 with weight $\sum_{j=1-L}^{0} \alpha_{j-i}$. And to each vertex $1 \le j \le R$ is added an edge from 0 with weight $\sum_{i=j}^{R} \alpha_i$.

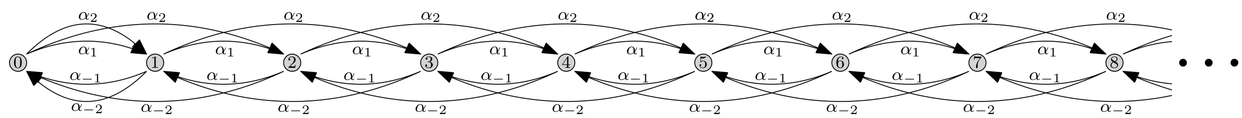

Figure 4.1. The graph $\mathcal{G}_+$.

The graph $\mathcal{G}_+$ has zero divergence at all sites except 0, where the divergence is $d^+ - d^- = \kappa_1$. Thus, in a sense there is a "net flow" of strength κ_1 from 0 to infinity, and to motivate the following lemma, the reader may imagine an edge "from ∞ to 0" with weight κ_1. In some sense, the following lemma extends Corollary 1.4.3 (1) to this infinite graph. One can prove it using a comparison between $\mathcal{G}$ and $\mathcal{G}_M$.

Lemma 4.2.3 ([22, Theorem 2])**.** *Under $P_{\mathcal{G}_+}$, $P_\omega^0(\tilde{T}_0 = \infty) \sim \mathrm{Beta}(\kappa_1, d^-)$.*

We will use Lemma 4.2.3 to prove Theorem 2.3.4 in two separate propositions. Recall that for a given walk $\mathbf{X}$ and integers $x < y$, the quantity $N_{x,y} = N_{x,y}(\mathbf{X})$ is defined as the number of trips from y to x.

Proposition 4.2.4. *Suppose $s \ge \kappa_1$. Then the following hold:*

1. $E_{\mathcal{G}}[E_\omega^0[N_0]^s] = \infty$.
2. *For all $x < y \in \mathbb{Z}$, $E_{\mathcal{G}}[E_\omega^0[N_{x,y}]^s] = \infty$.*

Like Theorem 2.3.3, this proposition not only gives a sufficient condition for zero speed, but gives the reason for zero speed. If $\kappa_0 \le 1$, the speed is zero because it takes a long time for the walk to exit small traps; on the other hand, if $\kappa_1 \le 1$, the speed is zero because the walk traverses large regions of $\mathbb{Z}$ many times.

Proof.

Outline

The philosophy of the proof is that the components of a Dirichlet random vector become more and more independent as their values become small. If (Y_1, Y_2, Y_3, Y_4) is a Dirichlet vector with parameters (a, b, c, d), and (X_1, X_2) and (X_3, X_4) are independent Dirichlet vectors with parameters (a, y) and (c, z), respectively, then $P(X_1 < \varepsilon, X_3 < \delta) \asymp P(Y_1 < \varepsilon, Y_3 < \delta)$ as $(\varepsilon, \delta) \to (0, 0)$. In other words, although these probabilities do not necessarily become approximately equal (even if $y = b$ and $z = c$), they are bounded by constant multiples of each other. For each $i = 1, \ldots, R$, the Dirichlet weight entering i from $[1 - R, 0]$ in $\mathcal{G}$ is the same as the Dirichlet weight entering i from 0 in $\mathcal{G}_+$. The goal of this proof is to exploit the comparability of small-value probabilities and perform a coupling between $(\omega(0, i))_{i=1}^R$ under $P_{\mathcal{G}_+}$ and $\left(\sum_{j=1-R}^0 \omega(j, i)\right)_{i=1}^R$ under $P_{\mathcal{G}}$.

Actually, we will couple vectors that distinguish between different edges to the same vertex. Let $E_0 := \{(i, j) : 1 - R \le i \le 0, 1 \le j \le i + R, \alpha_{j-i} > 0\}$ be the set of right-oriented edges in $\mathcal{G}$ that originate from or cross 0, and for every $e = (i, j) \in E_0$, let $\alpha_e = \alpha_{j-i}$. We consider random vectors $\mathbf{Z} = (Z_e)_{e \in E_0}$ and $\mathbf{Y} = (Y_e)_{e \in E_0}$, and a measure P' such that the distribution of $(Z_e)_{e \in E_0}$ under P' is the distribution of $(\omega(i, j))_{(i,j) \in E_0}$ under $P_{\mathcal{G}}$, and such that $\mathbf{Y}$ is a Dirichlet random vector with parameters $(\alpha_e)_{e \in E_0}$. The amalgamation property implies that $\left(\sum_{i=1-R}^0 Y_{(i,j)}\right)_{j=1}^R$ is distributed like $(\omega(0, j)_{j=1}^R$ under $P_{\mathcal{G}_+}$. The idea is to define a coupling event K, independent of $\mathbf{Y}$ and with positive probability, on which $Z_e \le Y_e$ for all e. We do not quite accomplish this, but we come close enough that we are able to use $\mathbf{Z}$ and $\mathbf{Y}$ to construct random environments ω_1 and ω_2, drawn respectively according to $P_{\mathcal{G}}$ and $P_{\mathcal{G}_+}$, such that on K, $\sum_{i=1-R}^0 P_{\omega_1}^i(\tilde{T}_{[1-R,0]} = \infty)$ is bounded above by a constant multiple of $P_{\omega_2}^0(\tilde{T}_0 = \infty)$. From Lemma 4.2.3 we get that $E_{\mathcal{G}_+}\left[\frac{1}{P_\omega^0(\tilde{T}_0 = \infty)}\right] = \infty$, so our coupling gives us $E'\left[\frac{1}{\sum_{i=1-R}^0 P_{\omega_1}^i(\tilde{T}_{[1-R,0]} = \infty)}\right] = \infty$. This is enough to give us $\mathbb{E}_{\mathcal{G}}^0[N_0] = \infty$, and more careful analysis yields $\mathbb{E}_{\mathcal{G}}^0[N_{x,y}] = \infty$.

Groundwork for the coupling

Suppose the edges in E_0 are enumerated as $e_1, \dots, e_k$ in some way. (In fact, we will enumerate them in a random way, yet to be described, but for now assume the enumeration is fixed.) We have said how $\mathbf{Z}$ and $\mathbf{Y}$ will be distributed under P'. By the amalgamation property, Z_{e_1} and Y_{e_1} are both beta random variables whose first parameter is α_{e_1}. Their second parameters may differ, but by (1.1) we nonetheless have $P'(Z_{e_1} < \varepsilon) \asymp \varepsilon^{\alpha_{e_1}} \asymp P'(Z_{e_1} < \varepsilon)$, where $f(\varepsilon) \asymp g(\varepsilon)$ means there exist positive constants c, C such that $cg(\varepsilon) \le f(\varepsilon) \le Cg(\varepsilon)$ for all $\varepsilon \in [0,1]$.

Note that for $1 \le i \le k-1$, $Y'_{e_i} := \frac{Y_{e_i}}{1-\sum_{j=1}^{i-1} Y_{e_j}} = \frac{Y_{e_i}}{\sum_{j=i}^{k} Y_{e_j}}$ is a beta random variable, independent of $Y_{e_1}, \dots, Y_{e_{i-1}}$, and with first parameter α_{e_i} (this comes from the restriction property, along with the amalgamation property). Let $Y'_{e_k} := \frac{Y_{e_k}}{1-\sum_{j=1}^{k-1} Y_{e_j}} = 1$. Likewise, for $1 \le i \le k$, $Z'_{e_i} := \frac{Z_{e_i}}{1-\sum_{j=1}^{i-1} Z_{e_j} \mathbb{1}_{\{\underline{e_j}=\underline{e_i}\}}}$ is a beta random variable, independent of $Z_{e_1}, \dots, Z_{e_{i-1}}$, and with first parameter α_{e_i}. (We do not have $Z_{e_k} = 1$ a.s., because Z_{e_k} corresponds to an edge from a vertex $i \le 0$ to a vertex $j > 0$, and there are still edges from i to vertices to the left of 0.) By (1.1), $P'(Z'_{e_i} < \varepsilon) \asymp \varepsilon^{\alpha_i} \asymp P'(Y'_{e_i} < \varepsilon)$, for $0 \le i \le k-1$. Thus, there exists a constant c such that for all $\varepsilon \in [0,1]$, and for all $i = 1, \dots, k-1$, we have $P'(Z'_{e_k} < \varepsilon) \ge cP'(Y'_{e_k} < \varepsilon)$. This c may depend on the chosen permutation $(e_1, \dots, e_k)$ of E_0, but there are only finitely many permutations, so we may assume c is small enough to work for any of them. For each $1 \le i \le k$, let $F_{Z'_{e_i}}$ be the cdf for Z'_{e_i}, and let $Q_{Z'_{e_i}}$ be the associated quantile function (since $F_{Z'_{e_i}}$ is continuous and strictly increasing on $[0,1]$, $Q_{Z'_{e_i}}$ is simply the inverse of $F_{Z'_{e_i}}$ restricted to the interval $[0,1]$). Similarly, for $1 \le i \le k$, let $F_{Y'_{e_i}}$ and $Q_{Y'_{e_i}}$ be the cdf and quantile function for Y'_{e_i}.[2] Choose ℓ to be an integer large enough that $\frac{1}{\ell} < c$. Then

$$F_{Z'_{e_i}} \ge \frac{1}{\ell} F_{Y'_{e_i}}, \quad 1 \le i \le k. \tag{4.12}$$

Now given $\mathbf{Y}' = (Y'_{e_1}, \dots, Y'_{e_k})$ and $\mathbf{Z}' = (Z'_{e_1}, \dots, Z'_{e_k})$, we can recover $\mathbf{Y}$ and $\mathbf{Z}$. First, $Y_{e_1} = Y'_{e_1}$ and $Z_{e_1} = Z'_{e_1}$, and then if Y_j and Z_j are known for $1 \le j \le i$, then the formulas for Y'_{e_i} and Z'_{e_i} can be used to find Y_{e_i} and Z_{e_i}. Therefore, one way to generate the vector $\mathbf{Y}$ is to

[2]↑Since Y'_{e_k} is identically 1, its cdf is 0 to the left of 1 and 1 at 1, and its quantile function is identically 1. All other Y'_{e_i} have continuous cdfs and quantile functions.

generate independent beta random variables $Y'_{e_1}, Y'_{e_2}, \ldots, Y'_{e_{k-1}}$ with appropriate parameters (which depend on the permutation $(e_1, \ldots, e_k)$) and then use these to recover $\mathbf{Y}$. Similarly, $\mathbf{Z}$ can be generated by means of independent beta random variables $Z'_{e_1}, \ldots, Z'_{e_k}$. Under this method, the chosen permutation $(e_1, \ldots, e_k)$ affects the parameters for the Y'_{e_i} and Z'_{e_i}, as well as the order in which they are put together, but the distributions of $\mathbf{Y}$ and $\mathbf{Z}$, are the same regardless of the chosen permutation.

The coupling

Our probability space is $[0,1]^k \times \{0, \ldots, \ell-1\}^k \times \Omega_{\mathbb{Z}}$. Let P' be the product measure whose marginals on $\Omega_{\mathbb{Z}}$ are equal to $P_{\mathcal{G}}$, and whose marginals on $[0,1]^k \times \{0, \ldots, \ell-1\}^k$ are uniform. An element of our probability space will be of the form

$$(U_1, \ldots, U_k, V_1, \ldots, V_k, \omega),$$

where the U_i take values on $[0,1]$, the V_i take integer values from 0 to $\ell - 1$, and ω can be any environment on $\mathbb{Z}$. Define the function $W_i := \frac{V_i + U_i}{\ell}$. Then the W_i are i.i.d. uniform $[0,1]$ under P'.

Recall that $\omega^{>0}$ is the environment ω to the right of 0; that is, if $\omega = (\omega(a,b))_{a,b\in\mathbb{Z}}$, then $\omega^{>0} = (\omega(a,b))_{a,b\in\mathbb{Z},a>0}$. The values of $P^i_\omega(T_0 = \infty)$ are determined by $\omega^{>0}$ for $1 \le i \le R$. For a given $\omega^{>0}$, let $(e_1, \ldots, e_k)$ be a permutation of E_0 such that

$$i < j \quad \Rightarrow \quad P^{\overline{e_j}}_\omega(T_0 = \infty) \le P^{\overline{e_i}}_\omega(T_0 = \infty). \tag{4.13}$$

To get such an arrangement, sort vertices $1 \le j \le R$ in order of $P^j_\omega(T_0 = \infty)$, then sort the edges $(i,j) \in E_0$ primarily according to the rank of j and secondarily according to the value of i.

We can now use uniform random variables along with quantile functions to get $\mathbf{Z}' = (Z'_{e_1}, \ldots, Z'_{e_k})$ and $\mathbf{Y}' = (Y'_{e_1}, \ldots, Y'_{e_k})$. Letting $Z'_{e_i} = Q_{Z'_{e_i}}(W_1)$ gives us the desired distribution for each Z'_{e_i} under P', and letting $Y'_{e_i} = Q_{Y'_{e_i}}(U_1)$ gives us the desired distribution for each Y'_{e_i}. This gives us $\mathbf{Y}'$ and $\mathbf{Z}'$, from which we may then recover $\mathbf{Z}$ and $\mathbf{Y}$. Even though the specific permutation $(e_1, \ldots, e_k)$ is used along with the U_i in defining $\mathbf{Y}$, the distribution of $\mathbf{Y}$ is the same for any fixed permutation, and so $\mathbf{Y}$ is independent of ω. Similarly, $\mathbf{Z}$ is also independent of ω.

Define the *coupling event* K to be the event that $V_1 = \ldots = V_k = 0$, the walk is transient to the right, P_ω–a.s., and $\omega(i,j) > 0$ iff $\alpha_{j-i} > 0$ for all $i,j \in \mathbb{Z}$. Because these last two conditions each have P' probability 1, K is independent of ω as well as $\mathbf{Y}$, and has positive probability $P'(K) = \left(\frac{1}{\ell}\right)^k$. On K, $W_i = \frac{1}{\ell}U_i$ for $1 \le i \le R$. Let $t \in [0,1]$. Then $Q_{Z'_{e_i}}(\frac{1}{\ell}t)$ is the unique x such that $F_{Z'_{e_i}}(x) = \frac{1}{\ell}t$. For this x, applying (4.12) gives us $\frac{1}{\ell}t \ge \frac{1}{\ell}F_{Y'_{e_i}}(x)$, or $t \ge F_{Y'_{e_i}}(x)$. Applying the increasing function $Q_{Y'_{e_i}}$ to both sides, we get $Q_{Y'_{e_i}}(t) \ge x = Q_{Z'_{e_i}}(\frac{1}{\ell}t)$. This is true for all $t \in [0,1]$, so on the event K, we have

$$Z'_{e_i} \le Y'_{e_i}, \qquad i = 1, \ldots, k. \tag{4.14}$$

We now describe how to use $\mathbf{Z}$, $\mathbf{Y}$, and ω to create environments ω_1 and ω_2, drawn according to $P_{\mathcal{G}}$ and $P_{\mathcal{G}_+}$, respectively. For $i,j > 0$, let $\omega_1(i,j) = \omega_2(i,j) = \omega(i,j)$. We also let $\omega_1(i,j) = \omega(i,j)$ for $i > 0$ and any $j \le 0$. But for ω_2, transition probabilities from positive to negative vertices are "collapsed" to 0. That is, for $i > 0$, we let $\omega_2(i,0) = \sum_{j \le 0} \omega(i,j)$. By the amalgamation property, one can check that transition probabilities at sites greater than 0 are drawn according to $P_{\mathcal{G}_+}$. Moreover, for $i \ge 1$, we have

$$P^i_{\omega_1}(T_{\le 0} = \infty) = P^i_{\omega_2}(T_0 = \infty) \tag{4.15}$$

Then let $\omega_2(0,j) = \sum_{i=1-R}^{0} Y_{(i,j)}$, $j = 1, \ldots, R$. By the distribution of $\mathbf{Y}$ and the fact that it is independent of ω, transition probabilities at sites greater than or equal to 0 of ω_2 are drawn according to $P_{\mathcal{G}_+}$ (sites less than 0 don't matter, so for example we can let $\omega_2(i,j) = \mathbb{1}_{\{j=i\}}$ whenever $i < 0$).

For $i \notin [1-R, 0]$, and for all j, let $\omega_1(i,j) = \omega(i,j)$. For $i \in [1-R, 0]$ and for all $j > 0$ with $(i,j) \in E_0$, let $\omega_1(i,j) = Z_{(i,j)}$. For all $i \in [1-R, 0]$ and $j \le 0$, we keep $\omega_1(i,j)$ the same as $\omega(i,j)$, but scaled to ensure that $\sum_j \omega_1(i,j) = 1$. That is,

$$\omega_1(i,j) = \left(1 - \sum_{r>0} Z_{(i,r)}\right) \frac{\omega(i,j)}{\sum_{r \le 0} \omega(i,r)}$$

Notice that under this definition, $\left(\frac{\omega_1(i,j)}{\sum_{r\leq 0}\omega_1(i,r)}\right)_{r\leq 0} = \left(\frac{\omega(i,j)}{\sum_{r\leq 0}\omega(i,r)}\right)_{r\leq 0}$. By the restriction property, $\left(\frac{\omega(i,j)}{\sum_{r\leq 0}\omega(i,r)}\right)_{r\leq 0}$ is independent of $(\omega(i,r))_{r>0}$ under $P_{\mathcal{G}}$, and by the way we have defined ω_1, $\left(\frac{\omega_1(i,j)}{\sum_{r\leq 0}\omega_1(i,r)}\right)_{r\leq 0}$ is independent of $(Z_{(i,r)})_{r>0}$ under P'. Therefore, since $(Z_{(i,r)})_{r>0} \overset{(L)}{=} (\omega(i,r))_{r>0}$ by construction, we have $(\omega_1(i,j))_j \overset{(L)}{=} (\omega(i,j))_j$ for each i, and the transition probability vectors are all independent. Hence the law of ω_1 is $P_{\mathcal{G}}$.

Comparing sums of probabilities

The goal for this part of the proof is to show that on the event K,

$$\sum_{i=1}^{k} Z_{e_i} P_{\omega_1}^{\overline{e_i}}(T_{\leq 0} = \infty) \leq C \sum_{i=1}^{k} Y_{e_i} P_{\omega_2}^{\overline{e_i}}(T_0 = \infty) \tag{4.16}$$

for some deterministic constant C. Note that the sum on the right side of (4.16) is equal to $P_{\omega_2}^{0}(\tilde{T}_0 = \infty)$.

By (4.15), we could get (4.16) by showing that $Z_{e_i} \leq CY_{e_i}$ for all i. However, achieving this precisely would require a more elaborate coupling. The difficulty is that if, for example, Y_{e_1} is very close to 1, all other Y_{e_i} are forced to be very small, whereas some of the Z_{e_i} are independent of Z_{e_1}. Our specific ordering of the edges $e_1, \ldots, e_k$ allows us to get around this difficulty.

Let r be the smallest integer in $\{1, \ldots, R\}$ such that $\sum_{i=1}^{r} Y_{e_i} > \frac{1}{2}$ (r is random). For $1 \leq i \leq r$, on the event K we have

$$Z_{e_i} < \frac{Z_{e_i}}{1 - \sum_{j=1}^{i-1} Z_{e_j} \mathbb{1}_{\{\underline{e_j} = \underline{e_i}\}}} \leq \frac{Y_{e_i}}{1 - \sum_{j=1}^{i-1} Y_{e_j}} \leq 2Y_{e_i}.$$

(The middle terms are the definitions of Z'_{e_i} and Y'_{e_i}, respectively.) We now have

$$\begin{aligned}
\sum_{i=1}^{k} Z_{e_i} P_{\omega_1}^{\bar{e}_i}(T_{\leq 0} = \infty) &= \sum_{i=1}^{r} Z_{e_i} P_{\omega_1}^{\bar{e}_i}(T_{\leq 0} = \infty) + \sum_{i=r+1}^{k} Z_{e_i} P_{\omega_1}^{\bar{e}_i}(T_{\leq 0} = \infty) \\
&\leq \sum_{i=1}^{r} 2Y_{e_i} P_{\omega_2}^{\bar{e}_i}(T_0 = \infty) + \sum_{i=r+1}^{k} P_{\omega_2}^{\bar{e}_i}(T_0 = \infty) \\
&\leq \sum_{i=1}^{r} 2Y_{e_i} P_{\omega_2}^{\bar{e}_i}(T_0 = \infty) + kP_{\omega_2}^{\bar{e}_r}(T_0 = \infty),
\end{aligned} \tag{4.17}$$

where, for the last line, we used the fact that $P_{\omega}^{\bar{e}_i}(T_0 = \infty)$ is non-increasing in i by (4.13). We want to combine the two terms from (4.17) into one. To do this, we note

$$\begin{aligned}\sum_{i=1}^{r} 2Y_{e_i} P_{\omega_2}^{\bar{e}_i}(T_0 = \infty) &\geq \sum_{i=1}^{r} 2Y_{e_i} P_{\omega_2}^{\bar{e}_r}(T_0 = \infty)\\ &= 2P_{\omega_2}^{\bar{e}_r}(T_0 = \infty)\sum_{i=1}^{r} Y_{e_i}\\ &\geq P_{\omega_2}^{\bar{e}_r}(T_0 = \infty),\end{aligned}$$

where we used the same non-increasing property for the first line, and the definition of r in the last line. Applying this to (4.17) gives us

$$\begin{aligned}\sum_{i=1}^{k} Z_{e_i} P_{\omega_1}^{\bar{e}_i}(T_{\leq 0} = \infty) &\leq \sum_{i=1}^{r} 2Y_{e_i} P_{\omega_2}^{\bar{e}_i}(T_0 = \infty) + k\sum_{i=1}^{r} 2Y_{e_i} P_{\omega_2}^{\bar{e}_i}(T_0 = \infty)\\ &= 2(k+1)\sum_{i=1}^{r} Y_{e_i} P_{\omega_2}^{\bar{e}_i}(T_0 = \infty)\\ &\leq 2(k+1)\sum_{i=1}^{k} Y_{e_i} P_{\omega_2}^{\bar{e}_i}(T_0 = \infty).\end{aligned} \tag{4.18}$$

This is exactly (4.16).

Comparing expectations

We consider the probability in ω_1, starting from a point a in $(-\infty, 0]$, of never hitting the set $[1-R, 0]$ at a positive time. If ω_1 is transient to the right and jumps to the right are bounded by R, then the only way for this to occur is for a to be in $[1-R, 0]$, for the first step to be to the right of 0, and then for the walk to never again hit a site to the left of 0. Thus, on the coupling event K,

$$\begin{aligned}
\max_{a\le 0} P^a_{\omega_1}(\tilde{T}_{[1-R,0]}=\infty) &\le \sum_{i=1-R}^{0} P^i_{\omega_1}(\tilde{T}_{[1-R,0]}=\infty) \\
&= \sum_{i=1-R}^{0}\sum_{j=1}^{R} Z_{(i,j)} P^j_{\omega_1}(T_{\le 0}=\infty) \\
&\le 2(k+1)\sum_{i=1-R}^{0}\sum_{j=1}^{R} Y_{(i,j)} P^j_{\omega_2}(T_0=\infty) \\
&= 2(k+1) P^0_{\omega_2}(\tilde{T}_0=\infty). \qquad (4.19)
\end{aligned}$$

It is straightforward to check by induction that for all $n \ge 1$, $a \le 0$,

$$P^a_{\omega_1}(N_{[1-R,0]} \ge n) \ge \min_{1-R\le i\le 0} P^i_{\omega_1}(\tilde{T}_{[1-R,0]}<\infty)^{n-1}. \qquad (4.20)$$

Summing over all $n \ge 1$ in (4.20) and applying (4.19), we get on the coupling event K,

$$\begin{aligned}
E^a_{\omega_1}[N_{[1-R,0]}] &= \sum_{n=1}^{\infty} P^a_{\omega_1}(N_{[1-R,0]} \ge n) \\
&\ge \sum_{n=1}^{\infty} \min_{1-R\le i\le 0} P^i_{\omega_1}(\tilde{T}_{[1-R,0]}<\infty)^{n-1} \\
&= \frac{1}{\max_{1-R\le i\le 0} P^i_{\omega_1}(\tilde{T}_{[1-R,0]}=\infty)} \\
&\ge \frac{1}{2(k+1)P^0_{\omega_2}(\tilde{T}_0=\infty)} \\
&= \frac{1}{2(k+1)} E^0_{\omega_2}[N_0]. \qquad (4.21)
\end{aligned}$$

Since the event K is independent of ω_2, we conclude that

$$\begin{aligned}
E'\left[E^0_{\omega_1}[N_{[1-R,0]}]^s\right] &\geq E'\left[E^0_{\omega_1}[N_{[1-R,0]}]^s \mathbb{1}_K\right] \\
&\geq E'\left[\frac{1}{2^s(k+1)^s} E^0_{\omega_2}[N_0]^s \mathbb{1}_K\right] \\
&= \frac{1}{2^s(k+1)^s} E'\left[E^0_{\omega_2}[N_0]^s\right] P'(K) \\
&= \frac{1}{\ell^k}\frac{1}{2^s(k+1)^s} E'\left[E^0_{\omega_2}[N_0]^s\right].
\end{aligned}$$

By the way ω_1 and ω_2 are distributed, this means

$$\begin{aligned}
E_{\mathcal{G}}\left[E^0_{\omega}[N_{[1-R,0]}]^s\right] &\geq \frac{1}{\ell^k}\frac{1}{2^s(k+1)^s} E_{\mathcal{G}_+}\left[E^0_{\omega}[N_0]^s\right] \\
&= \frac{1}{\ell^k}\frac{1}{2^s(k+1)^s} E_{\mathcal{G}_+}\left[\frac{1}{P^0_\omega(\tilde{T}_0 = \infty)^s}\right]
\end{aligned} \tag{4.22}$$

If $s \geq \kappa_1$, then the right side of (4.22) is infinite by Proposition 4.2.3 and (1.1), so we have

$$\begin{aligned}
\infty &= \mathrm{E}_{\mathcal{G}}[E^0_\omega[N_{[1-R,0]}]^s] \\
&= E_{\mathcal{G}}\left[\left(\sum_{i=1-R}^{0} E^0_\omega[N_i]\right)^s\right] \\
&\leq E_{\mathcal{G}}\left[\left(\sum_{i=1-R}^{0} E^i_\omega[N_i]\right)^s\right] \\
&\leq R^s E_{\mathcal{G}}\left[\left(\max_{1-R\leq i\leq 0} E^i_\omega[N_i]\right)^s\right] \\
&= R^s \sum_{i=1-R}^{0} E_{\mathcal{G}}\left[\left(E^i_\omega[N_i]\right)^s\right] \\
&= R^{s+1} E_{\mathcal{G}}\left[\left(E^0_\omega[N_0]\right)^s\right].
\end{aligned}$$

This proves the first part of the proposition. At this point, the reader interested only in characterizing ballisticity may skip the remainder of the proof, and may also skip Proposition 4.2.5, going straight to Section 4.2.3. However, the next part of this proof and Proposition 4.2.5 together provide an important insight into the behavior of the walk: namely, that the walk is expected to oscillate back and forth between any two points infinitely many times precisely when $\kappa_1 \leq 1$.

Arbitrarily large backtracking

We now want to prove the second part of the proposition, which strengthens our result to show that the expected number of oscillations between any two points is infinite. We do this via the following claim.

Claim 4.2.4.1. For any $a \leq 0$ and $x < y \leq 0$ we have

$$E_{\mathcal{G}}\left[E_\omega^a[N_{x,y}]^s \,\middle|\, \omega^{\leq -R}\right] = \infty, \;\; P_{\mathcal{G}}\text{–a.s.}$$

Assume for now that the claim is true. Taking expectations on both sides gives us

$$E_{\mathcal{G}}\left[E_\omega^a[N_{x,y}]^s\right] = \infty. \tag{4.23}$$

Let $x < y \in \mathbb{Z}$. If $y \leq 0$, then letting $a = 0$ in (4.23) gives us $E_{\mathcal{G}}\left[E_\omega^0[N_{x,y}]^s\right] = \infty$, which is exactly what we needed to show for the second part of the proposition. If $y > 0$, then (4.23) gives us $E_{\mathcal{G}}\left[E_\omega^{-y}[N_{x-y,0}]^s\right] = \infty$, and then the translation invariance of $\mathcal{G}$ gives us $E_{\mathcal{G}}\left[E_\omega^0[N_{x,y}]^s\right] = \infty$.

It remains, then, to prove the claim. Under P', ω_1 is drawn according to $P_{\mathcal{G}}$. Since ω_1 agrees with ω on $(-\infty, -R]$, our claim is equivalent to the statement $E'\left[E_{\omega_1}^a[N_{x,y}]^s \,\middle|\, \omega^{\leq -R}\right] = \infty$, P'–a.s. And since $\sigma(\omega^{\leq -R})$ is coarser than $\sigma(\omega^{\leq 0})$, it suffices to show that

$$E'\left[E_{\omega_1}^a[N_{x,y}]^s \,\middle|\, \omega^{\leq 0}\right] = \infty, \quad P'\text{–a.s.} \tag{4.24}$$

We first show that for $i \in [1-R, 0]$ and $j \leq 0$, there is a constant $C > 0$ such that on the event K,

$$\frac{\omega_1(i,j)}{\omega(i,j)} \geq C. \tag{4.25}$$

Recall that for $i \in [1-R, 0]$ and $j \le 0$, we have defined

$$\omega_1(i,j) = \left(1 - \sum_{r>0} Z_{(i,r)}\right) \frac{\omega(i,j)}{\sum_{r\le 0} \omega(i,r)}.$$

Now for $i \in [1-R, 0]$, we know that $\sum_{r>0} Z_{(i,r)}$ can be arbitrarily close to 1. However, we assert it is bounded away from 1 on the coupling event K, where necessarily $W_1, W_2, \ldots, W_k < \frac{1}{\ell}$. In fact, we assert that on this event, $\sum_{r>0} Z_{(i,r)}$ is maximized when $W_1 = \cdots = W_k = \frac{1}{\ell}$. One can check by induction that for a given $1-R \le i \le 0$,

$$\sum_{r>0} Z_{(i,r)} = 1 - \prod_{r>0}(1 - Z'_{(i,r)}). \tag{4.26}$$

Since all the $Z'_{(i,r)}$ are independent, the right side of (4.26) is maximized when they are all as large as possible. On K, the largest they can get is when all the W_i are equal to $\frac{1}{\ell}$, and this yields a value less than 1, proving our assertion and giving us (4.25).

Let $-M < 0$. For $1-R \le i \le 0$, consider the (i,x,y) *excursion event* $\mathcal{E}_{i,x,y}$ where:

- $X_0 = i$
- If $i \neq y$, the walk hits y before returning to i or leaving $(-\infty, 0]$.
- After T_y, the walk hits x without hitting i more than once[3] in between and without leaving $(-\infty, 0]$.

We say an (i,x,y) excursion event starts at time n if $(X_n, X_{n+1}, \ldots) \in \mathcal{E}_{i,x,y}$. Then the number of such excursion events for any i is no more than $2N_{x,y}$. This is because each trip from y to x can count toward at most two excursion events due to the requirement that there be only one visit to i in between visiting y and x.[4]

Fix $\omega^{\le 0}$. For any $i \le 0$, the probability under P^i_ω of any finite path that stays within $(-\infty, 0]$ is fixed. On the event K (which is independent of $\omega^{\le 0}$ and therefore still has

[3]↑It is necessary to allow for the walk to hit i once on the way to x in case $b > i$ and the only way to reach x from y is through i. For all other cases, we could require that the walk avoid i between hitting y and hitting x, but to avoid treating this case separately, we allow one visit to i in all cases, at the cost of a factor of 2.

[4]↑For example, if the walk goes from i to y to i again to y again and then to x, then an excursion event started at each of the times the walk was at i, but there was only one trip from to y to x.

probability $\frac{1}{\ell^k}$ conditioned on $\omega^{\leq 0}$), the probability under $P^i_{\omega_1}$ of such a path is bounded from below due to (4.25). Therefore, on the event K, there exists a positive constant $c = c(\omega^{\leq 0})$ such that on K,

$$\min_{1-R\leq i\leq 0} P^i_{\omega_1}(\mathcal{E}_{i,x,y}) > c. \tag{4.27}$$

(For each i consider a particular finite path that achieves $\mathcal{E}_{i,x,y}$, take c_i to be a lower bound for the probability under $P^i_{\omega_1}$ of taking that path, then take c to be the minimum of the c_i.)

We have from (4.21) that on K, $E^a_{\omega_1}[N_{[1-R,0]}] \geq \frac{1}{2(k+1)} E^0_{\omega_2}[N_0]$. Taking conditional expectations, we almost surely have

$$\begin{aligned}
E'\left[E^a_{\omega_1}[N_{[1-R,0]}]^s \,\middle|\, \omega^{\leq 0}\right] &\geq E'\left[E^a_{\omega_1}[N_{[1-R,0]}]^s \mathbb{1}_K \,\middle|\, \omega^{\leq 0}\right] \\
&\geq \frac{1}{2^s(k+1)^s} E'\left[E^0_{\omega_2}[N_0]^s \mathbb{1}_K \,\middle|\, \omega^{\leq 0}\right] \\
&= \frac{1}{2^s(k+1)^s} E'\left[E^0_{\omega_2}[N_0]^s\right] P'(K) \\
&= \infty,
\end{aligned}$$

where the first equality comes from the fact that $\omega^{\leq 0}$, K, and ω_2 are all independent. Now with probability 1,

$$\begin{aligned}
\infty &= E'\left[E^a_{\omega_1}[N_{[1-R,0]}]^s \,\middle|\, \omega^{\leq 0}\right] \\
&= E'\left[\left(\sum_{n=1}^{\infty} P^a_{\omega_1}(X_n \in [1-R,0])\right)^s \,\middle|\, \omega^{\leq 0}\right] \\
&= E'\left[\left(\sum_{i=1-R}^{0}\sum_{n=1}^{\infty} P^a_{\omega_1}(X_n = i)\right)^s \,\middle|\, \omega^{\leq 0}\right]
\end{aligned}$$

Multiplying by c^s, where c is the constant from (4.27), will not change the fact that the expression is infinite. And since c^s depends only on $\omega^{\leq 0}$, it may be pulled inside of an expectation conditioned on $\omega^{\leq 0}$. Therefore we almost surely have

$$\begin{aligned}
\infty &= E'\left[\left(\sum_{i=1-R}^{0}\sum_{n=1}^{\infty} cP_\omega^a(X_n = i)\right)^s \middle| \omega^{\leq 0}\right] \\
&\leq E'\left[\left(\sum_{i=1-R}^{0}\sum_{n=1}^{\infty} P_{\omega_1}^a(X_n = i)P_{\omega_1}^i(\mathcal{E}_{i,x,y})\right)^s \middle| \omega^{\leq 0}\right] \\
&= E'\left[\left(\sum_{i=1-R}^{0} E_{\omega_1}^a\left[\#\left\{n \in \mathbb{N}_0 : (X_k)_{k=n}^\infty \in \mathcal{E}_{i,x,y}\right\}\right]\right)^s \middle| \omega^{\leq 0}\right] \\
&\leq E'\left[\left(\sum_{i=1-R}^{0} E_{\omega_1}^a[2N_{x,y}]\right)^s \middle| \omega^{\leq 0}\right] \\
&= (2R)^s E'\left[E_{\omega_1}^a[N_{x,y}]^s \middle| \omega^{\leq 0}\right]
\end{aligned}$$

Thus, $E'\left[E_{\omega_1}^a[N_{x,y}]^s \middle| \omega^{\leq 0}\right] = \infty$ with probability 1. This is (4.24), which suffices to prove our claim, and with it our theorem. $\square$

We now prove a slightly strengthened converse to Proposition 4.2.4.

Proposition 4.2.5. *If $0 < s < \kappa_1$, then there is an $M \geq 0$ such that for all $x, y \in \mathbb{Z}$ with $y - x \geq M$, $E_{\mathcal{G}}\left[E_\omega^0[N'_{x,y}]^s\right] < \infty$.*

Proof. Let $0 < s < \kappa_1$. It suffices to find M such that for any $a \in \mathbb{Z}$,

$$E_{\mathcal{G}}\left[E_\omega^a[N'_{0,M}]^s\right] < \infty. \tag{4.28}$$

This is because for any x, y with $y - x = M$, (4.28) gives us

$$E_{\mathcal{G}}\left[E_\omega^{-x}[N'_{0,M}]^s\right] < \infty,$$

and then the shift-invariance of $\mathcal{G}$ gives us $E_{\mathcal{G}}\left[E_\omega^0[N'_{x,y}]^s\right] < \infty$. For $y - x > M$, we have $N'_{x,y} \leq N'_{x,x+M}$, which finishes the proof.

Let $M \geq 1$ and and fix $a \in \mathbb{Z}$. Suppose ω is such that jumps to the left and right are bounded by L and R, respectively, and the Markov chain is irreducible with almost-sure transience to the right (this is all true with $P_{\mathcal{G}}$-probability 1). Then if the walk traverses $[0, M]$ from right to left $n-1$ times, traversing the interval from right to left an nth time will require visiting one of the sites $M, \dots, M+L-1$ and then backtracking past 0. Now if there are at least $n-1$ traversals of $[0, M]$ from right to left, transience to the right implies that up to a P_ω^a-null set, there will be a next visit to $[M, M+L-1]$ after the $(n-1)$st such traversal. Let ℓ be the location of this visit if $n-1$ such traversals occur; otherwise, let ℓ be 0 (this value does not matter). On the event that there are $n-1$ traversals of $[0, M]$ from right to left,

$$\begin{aligned} P_\omega^a(N'_{0,M} \geq n \,|\, \ell) &= P_\omega^\ell(T_{\leq 0} < \infty) \\ &\leq \max_{0 \leq i \leq L-1} P_\omega^{M+i}(T_{\leq 0} < \infty). \end{aligned} \tag{4.29}$$

Taking conditional quenched expectations on both sides with respect to the event $N'_{0,M} \geq n-1$ (and noting that the right hand side of (4.29) is constant with respect to ω), we get

$$P_\omega^a\left(N'_{0,M} \geq n \,\middle|\, N'_{0,M} \geq n-1\right) \leq \max_{0 \leq i \leq L-1} P_\omega^{M+i}(T_{\leq 0} < \infty) \tag{4.30}$$

Hence we have

$$\begin{aligned} P_\omega^a(N'_{0,M} \geq n) &= P_\omega^a(N'_{0,M} \geq n-1) P_\omega^a(N'_{0,M} \geq n \,|\, N'_{0,M} \geq n-1) \\ &\leq P_\omega^a(N'_{0,M} \geq n) \max_{0 \leq i \leq L-1} P_\omega^{M+i}(T_{\leq 0} < \infty). \end{aligned}$$

From this, we can use induction to get

$$P_\omega^a(N'_{0,M} \geq n) \leq \left(\max_{0 \leq i \leq L-1} P_\omega^{M+i}(T_{\leq 0} < \infty) \right)^{n-1}.$$

Summing over all $n \geq 1$ gives us

$$E_\omega^a[N'_{0,M}] \leq \frac{1}{\min_{0 \leq i \leq L-1} P_\omega^{M+i}(T_{\leq 0} = \infty)}. \tag{4.31}$$

It therefore suffices to show that the right hand side has finite sth moment.

Notice that by our assumptions on ω, for any $x \geq 0$ and $y \geq x$, there is a z in $[y+1, y+R]$ such that $P_\omega^z(T_{\leq 0} = \infty) \geq P_\omega^x(T_{\leq 0} = \infty)$. This is because if a walk from x is to avoid backtracking to 0, it must enter $[y+1, y+R]$ before backtracking to 0, and then continue to avoid backtracking to 0. Thus, in every interval of length R to the right of 0, there is at least one site z (which depends on ω) such that $P_\omega^z(T_{\leq 0} = \infty) \geq \max_{1 \leq i \leq R} P_\omega^i(T_{\leq 0} = \infty)$. Call such a site an *escape site*, and let

$$\mathcal{E} = \mathcal{E}(\omega) := \{x \in [1, M] : x \text{ is an escape site}\}. \tag{4.32}$$

Then for $x > 0$,

$$\begin{aligned}
P_\omega^x(T_{\leq 0} = \infty) &\geq P_\omega^x(T_{\mathcal{E}} < T_{\leq 0}) \max_{1 \leq i \leq R} P_\omega^i(T_{\leq 0} = \infty) + P_\omega^x(T_{\mathcal{E}} = T_{\leq 0} = \infty) \\
&\geq P_\omega^x(T_{\mathcal{E}} < T_{\leq 0}) \max_{1 \leq i \leq R} P_\omega^i(T_{\leq 0} = \infty) + P_\omega^x(T_{\mathcal{E}} = T_0 = \infty) \max_{1 \leq i \leq R} P_\omega^i(T_{\leq 0} = \infty) \\
&= P_\omega^x(T_{\mathcal{E}} \leq T_{\leq 0}) \max_{1 \leq i \leq R} P_\omega^i(T_{\leq 0} = \infty).
\end{aligned}$$

Substituting this into (4.31) we get

$$\begin{aligned}
E_\omega^a[N'_{0,M}] &\leq \frac{1}{\min_{0 \leq i \leq L-1} P_\omega^{M+i}(T_{\mathcal{E}} \leq T_{\leq 0})} \cdot \frac{1}{\max_{1 \leq i \leq R} P_\omega^i(T_{\leq 0} = \infty)} \\
&= \frac{1}{1 - \max_{0 \leq i \leq L-1} P_\omega^{M+i}(T_{\leq 0} < T_{\mathcal{E}})} \cdot \frac{1}{\max_{1 \leq i \leq R} P_\omega^i(T_{\leq 0} = \infty)}
\end{aligned} \tag{4.33}$$

To show that this has finite sth moment for large enough M we will use Hölder's inequality. Choose s' such that $s < s' < \kappa_1$, and let $t = \frac{ss'}{s'-s}$. Thus, for any random variables X and Y, where $E[X^t] < \infty$ and $E[Y^{s'}] < \infty$, we will have $E[(XY)^s] < \infty$. We will show that the second term of the right hand side of (4.33) has finite s'th moment, and that the first term can have arbitrarily high finite moments for M sufficiently large, so that for large enough M, the first term has finite tth moment.

By arguing along the lines of Claim 3.2.1.1, we see that the distribution of $\max_{1 \leq i \leq R} P_\omega^i(T_0 = \infty)$ under $P_{\mathcal{G}_+}$ is the distribution of $\max_{1 \leq i \leq R} P_\omega^i(T_{\leq 0} = \infty)$ under $P_{\mathcal{G}}$. Recall from Lemma 4.2.3 that under $P_{\mathcal{G}_+}$, $P_\omega^0(\tilde{T}_0 = \infty) \sim \text{Beta}(\kappa_1, d^-)$. Now for $P_{\mathcal{G}_+}$–a.e. environment ω,

$P^0_\omega(\tilde{T}_0 = \infty) \le \max_{1\le i\le R} P^i_\omega(T_0 = \infty)$ by the Markov property, because $X_1 \in [1, R]$ $\mathbb{P}^0_{\mathcal{G}_+}$–a.s. We may conclude that

$$E_{\mathcal{G}}\left[\left(\frac{1}{\max_{1\le i\le R} P^i_\omega(T_{\le 0} = \infty)}\right)^{s'}\right] \le E_{\mathcal{G}_+}\left[\left(\frac{1}{P^0_\omega(\tilde{T}_0 = \infty)}\right)^{s'}\right] < \infty. \tag{4.34}$$

We now show that the first term of (4.33) has finite tth moment, provided M is chosen large enough. Let $A \in [0, L-1]$ be the maximizer in the denominator this term. That is, $M + A$ is the site within $[M, M+L-1]$ from which there is the highest probability of hitting 0 before hitting an escape site between 0 and M. For $i \in [0, L-1]$, let ω_i be the environment ω, modified at sites other than $M + i$ in $[M, \infty)$ so that the walk jumps from these sites to $M + i$ with probability 1 under ω_i. That is, for $y \ge M$, $\omega_i(y, M+i) = 1$.

Now under ω, a walk from any site to the right of $[M, M+L-1]$ must enter $[M, M+L-1]$ before hitting 0. By the strong Markov property, the site in $[M, \infty)$ with the best probability (under ω) of hitting 0 strictly before $\mathcal{E}$ is therefore A. Forcing the walk to jump from other sites in $[M, \infty)$ to site A can only increase the probability that the walk hits 0 before $\mathcal{E}$, by Lemma A.0.2. Therefore,

$$\max_{0\le i\le L-1} P^{M+i}_\omega(T_{\le 0} < T_{\mathcal{E}}) = P^{M+A}_\omega(T_{\le 0} < T_{\mathcal{E}}) \le P^{M+A}_{\omega_A}(T_{\le 0} < T_{\mathcal{E}}).$$

From this we get

$$\frac{1}{1 - \max_{0\le i\le L-1} P^{M+i}_\omega(T_{\le 0} < T_{\mathcal{E}})} \le \frac{1}{P^{M+A}_{\omega_A}(T_{\mathcal{E}} \le T_{\le 0})} \le \sum_{i=0}^{L-1} \frac{1}{P^{M+i}_{\omega_i}(T_{\mathcal{E}} \le T_{\le 0})}, \tag{4.35}$$

and it suffices to show that each term in the sum has finite tth moment for large enough M.

Say that a set $W \subseteq [1, M]$ is an *escape-type set* if W contains at least one element of every interval of length R contained in $[1, M]$. Then $\mathcal{E}$ is an escape-type set. Now for each escape-type set W, consider an environment $\omega_{i,W}$ such that:

1. All sites $y \in W(\omega)$ are sinks: for all $y \in W,\ z \in \mathbb{Z}$, $\omega_{i,W}(y, z) = \mathbb{1}_{\{z=y\}}$;

2. For $y = 1, \ldots L-1$, $\omega_{i,W}(y, 0) = \sum_{z\le 0} \omega_i(y, z)$ and for $z < 0$, $\omega_{i,W}(y, z) = 0$;

3. For all z, $\omega_{i,W}(0,z) = \mathbb{1}_{\{z=M+i\}}$;

4. All other transition probabilities are the same in $\omega_{i,W}$ as in ω_i.

By construction, $P^{M+i}_{\omega_i}(T_{\mathcal{E}} \leq T_{\leq 0}) = P^0_{\omega_{i,\mathcal{E}}}(\tilde{T}_0 = \infty)$. Hence

$$\frac{1}{P^{M+i}_{\omega_i}(T_{\mathcal{E}} \leq T_{\leq 0})} = E^0_{\omega_{i,\mathcal{E}}}[N_0]. \tag{4.36}$$

We wish to show, with Tournier's lemma, that this quantity has finite tth moment for sufficiently large M. Since $\mathcal{E}$ is random, $\omega_{i,\mathcal{E}}$ is not a Dirichlet environment, because the set $\mathcal{E}$ of sink sites is random. Nevertheless, for a fixed M, there are finitely many possible escape sets. Hence it suffices to show that for large enough M, $E^0_{\omega_{i,W}}[N_0]$ has finite tth moment for every escape-type set W. Note that sites outside of the set $[0, M+i+R]$ are unreachable from sites inside the set under $\omega_{i,W}$. By the amalgamation property, the restriction of $\omega_{i,W}$ to $[0, M+i+R]^2$ is distributed as a Dirichlet environment on a graph $\mathcal{G}_{M,i,W}$ with vertex set $[0, M+i+R]$ that looks like $\mathcal{G}$ on these vertices except that:

1. Directed edges from sites $w \in W$ are removed and replaced with one self-loop at each such site;

2. For each $y = 1, \ldots L-1$, all directed edges from y to sites less than or equal to 0 are replaced with one directed edge to 0 with the sum of their weights (in our illustration we use multiple edges for visual clarity);

3. All directed edges from 0 are replaced with one directed edge to $M+i$;

4. All directed edges from each $y \in [M+L, M+i+R]$ are replaced with one edge to $M+i$.

When there is only one edge from a vertex, its weight does not matter—weight 1 is as good as any. Figure 4.2 illustrates an example of a graph $\mathcal{G}_{M,i,W}$. Any strongly connected set S of vertices containing 0 must contain $M+i$ and a path from $M+i$ to 0, but cannot contain any vertices in W. Dividing $[0, M]$ into consecutive intervals of length m_0 (where $m_0 \geq \max(L, R)$ is large enough that every interval of length m_0 is strongly connected in

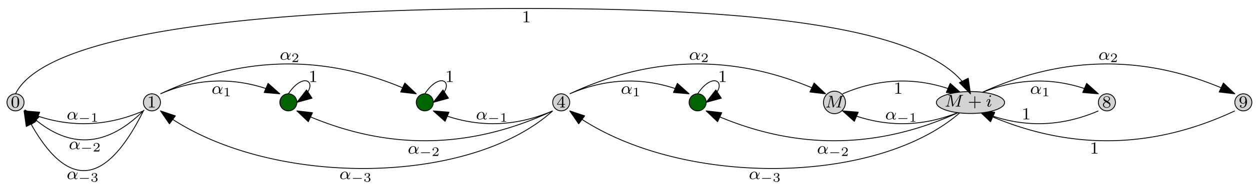

Figure 4.2. An example of the graph $\mathcal{G}_{M,i,W}$ with $L = 3$, $R = 2$, $M = 6$, $i = 1$, and $W = \{2, 3, 5\}$.

$\mathcal{G}$), we see that every such interval contains a vertex in S (because a path from $M+i$ to 0 in $\mathcal{G}_{i,W}$ cannot jump over L vertices), and every such interval contains a vertex in W (because such a vertex exists in every interval of length R). Hence, by the definition of m_0, every such interval must contain an edge from S to a vertex not in S. If $M \geq qm_0$, so that $[0, M]$ contains at least q disjoint intervals of length m_0, and if ε is the smallest weight any edge in $\mathcal{G}$ has, then $\beta_S \geq q\varepsilon$. Taking M sufficiently large raises this lower bound above t. Fix this large M. Then Tournier's lemma ensures that $E_{\mathcal{G}_{M,i,W}}[E_\omega^0[N_0]^t] < \infty$ for all escape-type sets W, which implies that $E_{\mathcal{G}}\left[E_{\omega_{i,W}}^0[N_0]^t\right] < \infty$. Since M is fixed, there are finitely many such W, so

$$E_{\mathcal{G}}\left[E_{\omega_{i,\varepsilon}}^0[N_0]^t\right] \leq E_{\mathcal{G}}\left[\sum_W E_{\omega_{i,W}}^0[N_0]^t\right] < \infty,$$

where the sum is taken over all escape-type sets W. Now, by (4.36), each term of the sum on the right hand side of (4.35) has finite tth moment, giving finite tth moment to the left hand side. This is what we needed to complete the proof, as we may now apply Hölder's inequality to (4.33) to see that $E_{\mathcal{G}}\left[E_\omega^a[N'_{0,M}]^s\right] < \infty$, which is precisely (4.28). $\square$

We have now proven each part of Theorem 2.3.4, which we recall here:

Theorem (Theorem 2.3.4)**.** *Let $\kappa_1 > 0$, so that the walk is transient to the right. Then, if $s > 0$, the following are equivalent:*

(a) $\kappa_1 > s$.

(b) There is an $M \geq 0$ such that for all $x, y \in \mathbb{Z}$ with $y - x \geq M$, $E_{\mathcal{G}}\left[E_\omega^0[N'_{x,y}]^s\right] < \infty$.

(c) There exist $x < y \in \mathbb{Z}$ such that $E_{\mathcal{G}}\left[E_\omega^0[N_{x,y}]^s\right] < \infty$.

Proof of Theorem 2.3.4.

$(a) \Rightarrow (b)$ This is Proposition 4.2.5.

$(b) \Rightarrow (c)$ This follows from the fact that $N'_{-M,0} \geq N_{-M,0}$.

$(c) \Rightarrow (a)$ This is the contrapositive of Proposition 4.2.4 (2). □

4.2.3 Using κ_0 and κ_1 to characterize ballisticity

From Theorems 2.3.3 and 2.3.4, we can conclude that if $s \geq \kappa_0$, then $E_{\mathcal{G}}[E_\omega^0[N_0]^s] = \infty$ due to finite trapping effects, and that if $s \geq \kappa_1$, then $E_{\mathcal{G}}[E_\omega^0[N_0]^s] = \infty$ due to large-scale backtracking effects. On the other hand, if $s < \min(\kappa_0, \kappa_1)$, then neither effect, on its own, is enough to cause $E_{\mathcal{G}}[E_\omega^0[N_0]^s]$ to be infinite. However, we must consider the possibility that the two effects could "conspire together", since the quenched probability of backtracking and hitting 0 is not completely independent of the quenched probability of hitting 0 a large number of times before exiting a small region. Nevertheless, we are able to show that there is enough independence that this is not an issue; $E_\omega^0[N_0]$ indeed has finite moments up to the minimum of κ_0 and κ_1. In this subsection, we prove Theorem 2.3.5, which we now recall.

Theorem (Theorem 2.3.5)**.** *Assume $\kappa_1 > 0$. Then $E_{\mathcal{G}}\left[(E_\omega^0[N_0])^s\right] < \infty$ if and only if $s < \min(\kappa_0, \kappa_1)$.*

Before proving it, we begin with the following lemma.

Lemma 4.2.6. *For any $z > 0$,*

$$E_{\mathcal{G}}\left[\frac{1}{P_\omega^0(T_{\geq z} < \tilde{T}_0)^s}\right] < \infty \quad \Leftrightarrow \quad s < \min(\kappa_0, d^+).$$

Equivalently, $E_{\mathcal{G}}\left[E_\omega^0\left[N_0^{(-\infty,z)}\right]^s\right] = \infty$ if and only if $s < \min(\kappa_0, d^+)$.

Proof. First, assume $s < \min(\kappa_0, d^+)$. We will consider an integer $-M < -R$. For an environment ω, let $j(\omega)$ be the vertex j in $[1-M, R-M]$ that maximizes $P_\omega^j(T_0 < T_{\geq z})$. Consider an environment ω' such that:

1. for $i \in [z-R, z-1]$, $\omega'(i,z) = \sum_{j \geq z} \omega(i,j)$ and for $j > z$, $\omega'(i,j) = 0$;

2. z is a sink—for all x, $\omega'(z,x) = \mathbb{1}_{\{x=z\}}$;

3. all other transition probabilities are the same in ω' as in ω.

By construction, $P^x_{\omega'}(\tilde{T}_0 < \infty) = P^x_{\omega'}(\tilde{T}_0 < T_z) = P^x_{\omega}(\tilde{T}_0 < T_{\geq z})$ for all x. We now modify the environment further. For each $1 \leq \ell \leq R$, let ω_ℓ be the environment such that

1. for $i \in [-M-L+1, -M]$, $\omega'(i,j) = \mathbb{1}_{\{j=\ell\}}$ for all j;

2. All other transition probabilities are the same in ω_ℓ as in ω'.

In particular, we are interested in $\omega_{j(\omega)}$. Because $j(\omega)$ maximizes $P^j_\omega(T_0 < T_{\geq z})$ on $[-M-L+1, -M]$, modifying sites in this set to send the walk directly to $j(\omega)$ can only increase the probability, from any starting point, that $T_0 < T_z$ (see Lemma A.0.2 for details). Thus,

$$P^0_{\omega_{j(\omega)}}(\tilde{T}_0 < T_{\geq z}) \leq P^0_{\omega'}(\tilde{T}_0 < T_{\geq z}) = P^0_{\omega}(\tilde{T}_0 < T_{\geq z}).$$

Moreover, for $P_{\mathcal{G}}$–a.e. ω, we have $E^0_{\omega_{j(\omega)}}[N_0] = \frac{1}{P^0_{\omega_{j(\omega)}}(\tilde{T}_0 = \infty)}$. It follows that

$$\frac{1}{P^0_\omega(T_{\geq z} < \tilde{T}_0)} \leq E^0_{\omega_{j(\omega)}}[N_0].$$

It suffices, therefore, to show that $E_{\mathcal{G}}[E_{\omega_{j(\omega)}}[N_0]^s] < \infty$. We would like to use Tournier's lemma. Although $\omega_{j(\omega)}$ is not distributed according to a Dirichlet distribution (since $j(\omega)$ is random), the amalgamation property implies that each ω_ℓ is distributed as a Dirichlet environment. In particular, the restriction of ω_ℓ to $[-M-L+1, z]^2$ is distributed as a Dirichlet environment on a graph $\mathcal{G}_\ell$ with vertices $[-M-L+1, z]$ and the following properties:

1. each vertex in $[-M-L+1, -M]$ has one edge to ℓ with arbitrary weight, say 1;

2. vertices in $[1-M, z-R-1]$ have the same edges with the same weights as in $\mathcal{G}$;

3. vertices in $[z-R, z-1]$ have the same edges with the same weights as in $\mathcal{G}$, except that edges that would terminate to the right of z terminate at z;[5]

[5]↑As usual, if this would result in multiple edges, they are collapsed into one edge with the sum of their weights, but we leave multiple edges in our illustration for clarity.

4. z has one self-loop with arbitrary weight, say 1.

With probability 1, no vertices to the left of $-M-L+1$ or to the right of z are reachable from $[-M, z]$, so what happens at these vertices does not really matter. Figure 4.3 shows an example of the graph $\mathcal{G}_\ell$.

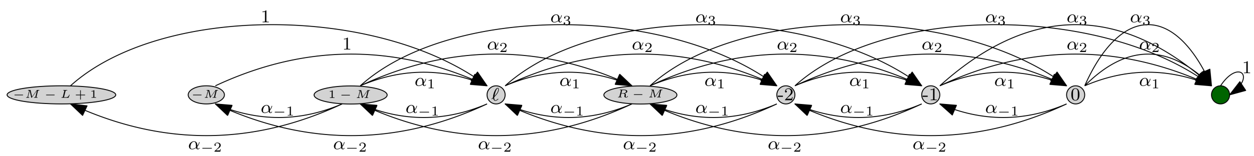

Figure 4.3. An example of the graph $\mathcal{G}_\ell$, where $L = 2$, $R = 3$, $z = 1$, $M = 6$, and $\ell = 2 - M$.

Moreover, since $1 - M \le j(\omega) \le R - M$, we have $E^0_{\omega_{j(\omega)}}[N_0] \le \sum_{\ell=1-M}^{R-M} E^0_{\omega_\ell}[N_0]$. Thus, to show that $E_{\mathcal{G}}\left[\frac{1}{P^0_\omega(T_{\ge z} < \tilde{T}_0)^s}\right] < \infty$, it suffices to show that $E_{\mathcal{G}}[E^0_{\omega_\ell}[N_0]^s] = E_{\mathcal{G}_\ell}[E^0_\omega[N_0]^s]$ is finite for each $1 - M \le \ell \le R - M$.

We make the following claim.

Claim 4.2.6.1. If M is chosen large enough, $\mathcal{G}_\ell$ will have the property that every finite, strongly connected set S of vertices with $0 \in S$ has $\beta_S \ge \min(d^+, \kappa_0)$.

The proof of this claim is very similar to the proof of Proposition 4.2.1, so we only sketch it here, making special note of important similarities to and differences from the earlier proof. Like $\mathcal{G}$, the graph $\mathcal{G}_\ell$ has the property that for any finite, strongly connected set set S of vertices, if $x < y$ are consecutive "non-insulated" vertices to the right of $-M$ that differ by more than m_0, then $(x, y) \subset S$. The graph $\mathcal{G}_\ell$ also has the property that every strongly connected set S' that contains m_0 or more consecutive vertices has total weight at least d^+ exiting S' to the right. However, because of how the graph is modified, there need not be any weight exiting S' to the left. We may therefore choose M large enough that every finite, strongly connected set S containing 0 either (a) does not reach $-M$, in which case S looks like a subset of $\mathcal{G}$ and $\beta_S \ge \kappa_0$, or (b) contains at least m_0 consecutive vertices in a row, in which case $\beta_S \ge d^+$, or (c) contains enough non-insulated vertices that $\beta_S \ge \min(d^+, \kappa_0)$. This completes the proof of the claim.

Since we have assumed $s < \min(\kappa_0, d^+)$, our claim gives us $s < \beta_S$ for all strongly connected sets S containing 0, and therefore Tournier's lemma tells us that $E_{\mathcal{G}_\ell}[E_\omega^0[N_0]^s] < \infty$.

This finishes one direction of the lemma, and the only direction that is needed for the rest of the paper. For the other direction, assume $s \geq \min(\kappa_0, d^+)$. If $s \geq \kappa_0$, then Theorem 2.3.3 implies $E_{\mathcal{G}}\left[E_\omega^0\left[N_0^{[-M,0]}\right]^s\right] = \infty$, which implies $E_{\mathcal{G}}\left[E_\omega^0\left[N_0^{(-\infty,z)}\right]^s\right] = \infty$. If $s \geq d^+$, then one can check that $E_{\mathcal{G}}\left[\left(\sum_{i<z,j\geq z} \omega(i,j)\right)^{-s}\right] = \infty$, and then arguments along the lines of the last part of the proof of Proposition 4.2.4 give the desired result. □

We are now ready to prove Theorem 2.3.5, which gives us the final piece for our characterization of ballisticity.

Proof of Theorem 2.3.5. The forward direction is implied by Theorem 2.3.3 and Theorem 2.3.4. For the reverse direction, let $s < s' < \min(\kappa_1, \kappa_0)$. We want to show that $E_{\mathcal{G}}\left[E_\omega^0[N_0]^s\right] < \infty$. Since, for all ω, $E_\omega^0[N_0] = \frac{1}{P_\omega^0(\tilde{T}_0 = \infty)}$, we must examine the quantity $P_\omega^0(\tilde{T}_0 = \infty)$.

Let $M = qm_0$ be a positive multiple of m_0 (we will later take q to be large enough to satisfy a given condition, but we will not take q to infinity). For each $1 \leq j \leq q$, let $B_j = ((j-1)m_0, jm_0]$, and define

$$f_\omega(B_j) := \min_{x \in B_j, 1 \leq i \leq R} P_\omega^x(X_{T_{B_j^c}} = jm_0 + i)$$

Since $f_\omega(B_j)$ only depends on ω^{B_j}, the $f_\omega(B_j)$ are i.i.d.

We will show that for some $a, C > 0$, $P_{\mathcal{G}}(f_\omega(B_1) < \varepsilon) \leq C\varepsilon^a$. For a given vertex x, let $\underline{\omega^x} = \min_{\alpha_{y-x}>0} \omega(x,y)$. Since intervals of length m_0 are strongly connected, there is a path in $\mathcal{G}$ from every $x \in B_1$ to every $m_0 + i$, $1 \leq i \leq R$, that uses only vertices in B_1, and each at most once (since the existence of a path with loops implies the existence of a path without loops). Therefore,

$$f_\omega(B_1) \geq \left(\prod_{x \in B_1} \underline{\omega^x}\right).$$

Now for each x,

$$\begin{aligned}P_{\mathcal{G}}\left(\underline{\omega^x} < \varepsilon\right) &\leq (R+L) \max_{\alpha_{y-x}>0} P_{\mathcal{G}}(\omega(x,y) < \varepsilon) \\ &\leq C\varepsilon^a\end{aligned}$$

for some constant $C > 0$, where $a = \min_{\alpha_i \neq 0} \alpha_i$. Now by [29, Lemma 9], there is some r such that $P_{\mathcal{G}}(\prod_{x\in B_1} \underline{\omega^x} \leq \varepsilon) \leq C'\varepsilon^a(-\ln\varepsilon)^r$ for all ε sufficiently small. Let $a' < a$; then for large enough C'', we have $P_{\mathcal{G}}(\prod_{x\in B_1} \underline{\omega^x} < \varepsilon) \leq C''\varepsilon^{a'}$, and therefore $P_{\mathcal{G}}(f_\omega(B_1) < \varepsilon) \leq C''\varepsilon^{a'}$

Let $M = m_0 q$ be large enough that $a'q > t := \frac{ss'}{s'-s}$. Since all the $f_\omega(B_j)$ are i.i.d., we have

$$P_{\mathcal{G}}\left(\max_{0\leq j\leq q-1} f_\omega(B_j) < \varepsilon\right) \leq (C''\varepsilon^{a'})^q. \tag{4.37}$$

Now let j_* be the maximizer of $f_\omega(B_j)$ over $0 \leq j \leq q-1$. By (4.37) and the choice of M, we have

$$E_{\mathcal{G}}\left[\left(\frac{1}{f_\omega(B_{j_*})}\right)^t\right] < \infty. \tag{4.38}$$

As in the proof of Proposition 4.2.5, we note that for any random variables X and Y, where $E[X^t] < \infty$ and $E[Y^{s'}] < \infty$, we will have $E[(XY)^s] < \infty$ by Hölder's inequality.

For $P_{\mathcal{G}}$–a.e. environment ω we have

$$P_\omega^0(\tilde{T}_0 = \infty) \geq P_\omega^0(T_{B_{j_*}} < \tilde{T}_0) f_\omega(B_{j_*}) \max_{1\leq i\leq R} P_\omega^{j_* m_0 + i}(T_{\leq j_* m_0} = \infty).$$

This is because one way for the walk to never return to 0 is for it to hit B_{j_*} before returning to 0, then once it is in B_{j_*}, to make its way to the vertex $j_*m_0 + i$ just to the right of B_{j_*} that maximizes the probability of never backtracking to j_*m_0, and then to avoid backtracking to j_*m_0. Thus

$$\begin{aligned}\frac{1}{P_\omega(\tilde{T}_0=\infty)} &\le \frac{1}{P_\omega^0(T_{B_{j_*}}<\tilde{T}_0)f_\omega(B_{j_*})\max_{1\le i\le R}P_\omega^{j_*m_0+i}(T_{\le j_*m_0}=\infty)} \\ &= \sum_{j=1}^{q}\frac{\mathbb{1}_{\{j=j_*\}}}{P_\omega^0(T_{B_j}<\tilde{T}_0)f_\omega(B_{j_*})\max_{1\le i\le R}P_\omega^{jm_0+i}(T_{\le jm_0}=\infty)} \\ &\le \frac{1}{f_\omega(B_{j_*})}\sum_{j=1}^{q}\frac{1}{P_\omega^0(T_{B_j}<\tilde{T}_0)}\cdot\frac{1}{\max_{1\le i\le R}P_\omega^{jm_0+i}(T_{\le jm_0}=\infty)} \qquad (4.39)\end{aligned}$$

Now for each fixed j, $P_\omega^0(T_{B_j}<\tilde{T}_0)$ and $\max_{1\le i\le R}P_\omega^{jm_0+i}(T_{\le jm_0}=\infty)$ are independent. Under $P_{\mathcal{G}}$, the reciprocal of the former has finite s'th moment by Lemma 4.2.6, since $s'\le\min(\kappa_0,\kappa_1)\le\min(\kappa_0,d^+)$.

We show that the reciprocal of $\max_{1\le i\le R}P_\omega^{jm_0+i}(T_{\le jm_0}=\infty)$ has finite s'th moment. Under $P_{\mathcal{G}}$, its distribution is the same as that of $\max_{1\le i\le R}P_\omega^{i}(T_{\le 0}=\infty)$, which (arguing as in Claim 3.2.1.1) is also the distribution of $\max_{1\le i\le R}P_\omega^{i}(T_0=\infty)$ under $P_{\mathcal{G}_+}$. Now for $P_{\mathcal{G}_+}$–a.e. environment ω, $P_\omega^0(\tilde{T}_0=\infty)\le\max_{1\le i\le R}P_\omega^i(T_0=\infty)$. Hence

$$E_{\mathcal{G}}\left[\frac{1}{\max_{1\le i\le R}P_\omega^{jm_0+i}(T_{\le jm_0}=\infty)^{s'}}\right]\le E_{\mathcal{G}_+}\left[\frac{1}{P_\omega^0(\tilde{T}_0=\infty)^{s'}}\right]. \qquad (4.40)$$

Lemma 4.2.3 tells us that under $P_{\mathcal{G}_+}$, $P_\omega^0(\tilde{T}_0=\infty)\sim\mathrm{Beta}(\kappa_1,d^-)$. Since $s'\le\kappa_1$, the right side of (4.40) is finite by (1.1).

Returning now to (4.39), we have seen that the two fractions inside the sum are independent and each have finite s'th moment. Thus, the entire sum has finite s'th moment. By this and by (4.38), we may apply Hölder's inequality to get

$$
\begin{aligned}
E_{\mathcal{G}}[E_\omega^0[N_0]^s] &= E_{\mathcal{G}}\left[\frac{1}{P_\omega(\tilde{T}_0=\infty)^s}\right] \\
&\leq E_{\mathcal{G}}\left[\frac{1}{f_\omega(B_{j_*})^s}\left(\sum_{j=1}^{q}\frac{1}{P_\omega^0(T_{B_j}<\tilde{T}_0)}\cdot\frac{1}{\max_{v\in B_j}P_\omega^v(X_n>B_j \text{ for all } n\geq 1)}\right)^s\right] \\
&\leq E_{\mathcal{G}}\left[\left(\frac{1}{f_\omega(B_{j_*})^s}\right)^{\frac{s'}{s'-s}}\right]^{\frac{s'-s}{s'}} \\
&\qquad\cdot E_{\mathcal{G}}\left[\left(\sum_{j=1}^{q}\frac{1}{P_\omega^0(T_{B_j}<\tilde{T}_0)}\cdot\frac{1}{\max_{v\in B_j}P_\omega^v(X_n>B_j \text{ for all } n\geq 1)}\right)^{s'}\right]^{\frac{s}{s'}} \\
&< \infty.
\end{aligned}
$$

□

We are now able to completely characterize ballisticity.

Theorem (Theorem 2.3.6)**.** *The walk is ballistic if and only if* $\min(\kappa_0, |\kappa_1|) > 1$.

Proof of Theorem 2.3.6. By symmetry, we may assume $\kappa_1 > 0$ without loss of generality. Let $s = 1$ in Theorem 2.3.5, and then apply Lemma 2.3.2. □

5. ACCELERATION IN ONE DIMENSION

Recall from Chapter 2 that a *Bouchet acceleration function* on $\mathbb{Z}$ is a measurable function $\mathcal{A}$ from the space $\Omega_{\mathbb{Z}}$ of environments on $\mathbb{Z}$ to the space of distributions of $\mathbb{R}^+$-valued random variables, where $\mathcal{A}(\omega)$ only depends the environment ω within a finite distance from the origin. For an environment ω on $\mathbb{Z}$, a point $x \in \mathbb{Z}$, and a Bouchet acceleration function $\mathcal{A}$, we let $P^x_{\omega,\mathcal{A}}$ be the law of a continuous-time Markov chain that begins at x, moves in a sequence of steps according to P^x_ω, but with independent waiting times at each site y distributed according to $\mathcal{A}(\theta^y\omega)$. Then we let $\mathbb{P}^x_{\mathcal{G},\mathcal{A}} := P_{\mathcal{G}} \times P^x_{\omega,\mathcal{A}}$. By Proposition 5.1.1, there is a $\mathbb{P}^0_{\mathcal{G},\mathcal{A}}$–almost sure limiting speed $v(\mathcal{A})$. The main theorem of this chapter is the following.

Theorem (Theorem 2.4.1)**.** *Assume $\kappa_1 \neq 0$. Then $P_{\mathcal{G}}$ has essential slowing if and only if $|\kappa_1| \leq 1$.*

5.1 Redoing proofs with acceleration

We redo some of the work from Chapter 4 in the setting of accelerated random walks.

For a continuous-time walk $\mathbf{X} = (X_t)_{t\geq 0}$ on a vertex set V with vertex x, $N_x(\mathbf{X}) = \int_0^\infty \mathbb{1}_{X_t=x} dt$ is the amount of time the walk spends at site x. We likewise give N^S_x, T_x, T_S and so on definitions analogous to the discrete-time definitions, but with $\mathbb{N}_0$ replaced with $\mathbb{R}^{\geq 0}$, etc. See Appendix C for details.

Recall that for a given Bouchet acceleration function $\mathcal{A}$, a given environment ω, and a given $x \in \mathbb{Z}$, the measure $P^x_{\omega,\mathcal{A}}$ is defined to be the law of a continuous-time random walk whose path obeys the law P^x_ω, with independent wait times at each site y distributed according to $\mathcal{A}(\theta^y\omega)$. Define $\mathbb{P}^x_{\mathcal{A}} := P \times P^x_{\omega,\mathcal{A}}$, and let $v(\mathcal{A})$ be the $\mathbb{P}^0_{\mathcal{A}}$–a.s. limiting velocity of the walk. We first show that it exists.

We prove a generalization of 4.1.1 for accelerated, transient RWRE on $\mathbb{Z}$.

Proposition 5.1.1. *Let $\mathcal{A}$ be a Bouchet acceleration function, and let M be such that $\mathcal{A}(\omega)$ depends only on $\omega^{[-M,M]}$. Let P be a probability measure on $\Omega_{\mathbb{Z}}$ satisfying (C1), (C2), (C3), and (C4). Then the following hold:*

1. *There is a $\mathbb{P}^0_{\mathcal{A}}$–almost sure limiting velocity*

$$v(\mathcal{A}) := \lim_{n\to\infty} \frac{X_t}{t} = \frac{\mathbb{E}^0_{\mathcal{A}}[X_{\tau_2} - X_{\tau_1}]}{\mathbb{E}_{\mathcal{A}}[\tau_{M+2} - \tau_{M+1}]}, \tag{5.1}$$

   where the numerator is always finite, and the fraction is understood to be 0 if the denominator is infinite.

2. $\lim_{x\to\infty} \frac{T_{\geq x}}{x} = \frac{1}{v(\mathcal{A})}$*, where* $\frac{1}{v(\mathcal{A})}$ *is understood to be* ∞ *if* $v(\mathcal{A}) = 0$*.*

The proof follows [35] in defining *regeneration times* $(\tau_k)_{k=0}^{\infty}$. Let $\tau_0 := 0$, and for $k \geq 1$, we define

$$\tau_k := \min\{t \geq \tau_{k-1} : X_t > X_r \text{ for all } r < t, X_t \leq X_r \text{ for all } r > t\}. \tag{5.2}$$

A crucial fact is that the sequences $(X_{\tau_n} - X_{\tau_{n-1}})_{n=2}^{\infty}$ and $(\tau_n - \tau_{n-1})_{n=M+2}^{\infty}$ are stationary and ergodic. Using these regeneration times, we are able to derive a formula for $v(\mathcal{A})$, as well as a characterization in terms of hitting times.

It is standard (see, for example, [35], [37]) to prove a LLN in (1) for the discrete-time case (where we may take $M = 0$) by the following steps:

(a) Show that $\frac{X_{\tau_k}}{k}$ approaches $\mathbb{E}[X_{\tau_2} - X_{\tau_1}]$

(b) Show that $\frac{\tau_k}{k}$ approaches $\mathbb{E}[\tau_2 - \tau_1]$

(c) Show that $\mathbb{E}[X_{\tau_2} - X_{\tau_1}] < \infty$

(d) Conclude that the limit (5.1) holds for the subsequence $\left(\frac{X_{\tau_k}}{\tau_k}\right)_k$

(e) Use straightforward bounds that come from the definitions of the τ_k to get the limit for the entire sequence $\left(\frac{X_n}{n}\right)_n$.

Part (2) then follows from a comparison of $\frac{x}{H_{\geq x}}$ with a subsequence of $\frac{X_n}{n}$.

In the discrete-time case, the definition of the regeneration times is precisely set up so that both the sequences $(\tau_k - \tau_{k-1})_{k\geq 2}$ and $(X_{\tau_k} - X_{\tau_{k-1}})_{k\geq 2}$ are i.i.d. sequences, so proving the limits (a) and (b) is a matter of tracing how the i.i.d. feature follows from the definitions and applying the strong law of large numbers. In fact, arguing as in [35, Lemma 1], one can show that the triples

$$\xi_n := \Big(\tau_n - \tau_{n-1}\ ,\ (X_{\tau_{n-1}+i} - X_{\tau_{n-1}})_{i=1}^{\tau_n-\tau_{n-1}}\ ,\ (\omega^x)_{x=X_{\tau_{n-1}}}^{X_{\tau_n}-1}\Big) \tag{5.3}$$

are i.i.d. under $\mathbb{P}^0 = P \times P_\omega^0$ for $n \geq 2$. In the continuous-time case, however, where the definition is

$$\xi_n := \Big(\tau_n - \tau_{n-1}\ ,\ (X_{\tau_{n-1}+r} - X_{\tau_{n-1}})_{0\leq r\leq \tau_n-\tau_{n-1}}\ ,\ (\omega^x)_{x=X_{\tau_{n-1}}}^{X_{\tau_n}-1}\Big)\,, \tag{5.4}$$

there is some dependence, because the quenched distributions of the jump times can depend on the environment outside of $[X_{\tau_{n-1}}, X_{\tau_n} - 1]$. For the same reason, the sequence does not become stationary until $n \geq M + 2$, when the jumping times are guaranteed to only depend on the environment at or to the right of X_{τ_1}. Nevertheless, the $(\xi_n)_{n=M+2}^\infty$ are still stationary and ergodic under $\mathbb{P}_{\mathcal{A}}^0$ (and, in fact, finite-range dependence holds).

To show that the sequence $(\xi_n)_{n=M+2}^\infty$ is stationary and ergodic under the measure $\mathbb{P}_{\mathcal{A}}^0$, we generate a sequence $(\xi_n^*)_{n=1}^\infty$ under the measure $\mathbb{P}^0$. This determines $\omega^{\geq 0}$, which is enough to determine $\mathcal{A}(\theta^x\omega)$ for every $x \geq M$. By expanding the probability space, we can then generate random jumping times for each step to create a sequence $(\xi_n)_{n=M+2}^\infty$ drawn according to the measure $\mathbb{P}_{\mathcal{A}}$. Now given the sequence $(\xi_n^*)_{n=2}^\infty$, the distribution of each ξ_n depends only on $(\xi_k^*)_{k=n-M}^{n+M}$, since this is enough to determine ω^k for every $X_{\tau_n-1} - M \leq k < X_{\tau_n} + M$, which determines jump time distributions at each site k for $X_{\tau_n-1} \leq k < X_{\tau_n}$. It follows that $(\xi_n)_{n=M+2}^\infty$ is stationary and ergodic.

Therefore, by Birkhoff's ergodic theorem one may derive the same limits (a) and (b). The finiteness in (c) relates only to the path of the walk, not to the time taken, and therefore the proof is entirely unchanged from the proof given in Appendix A.Then (d) and (e) easily follow.

The main result of this subsection is the following.

Proposition 5.1.2. *Let P be a probability measure on $\Omega_{\mathbb{Z}}$ satisfying (C1), (C2), (C3), and (C4). Let $\mathcal{A}$ be a Bouchet acceleration function. Then we have the following:*

1. If $\mathbb{E}^0_{\mathcal{A}}[N_0] < \infty$, then $v(\mathcal{A}) > 0$.

2. If $\mathbb{E}^0[N_0] < \infty$, then $v > 0$.

Jump to proof.

The second statement is a special case of the first. For the rest of this section, assume P satisfies (C1), (C2), (C3), and (C4). We also use regeneration times to derive the following lemma.

Lemma 5.1.3. *For any $a, c \in \mathbb{Z}$,*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{k=c}^{x} N_k = \frac{1}{v(\mathcal{A})}, \ \mathbb{P}^a_{\mathcal{A}} - a.s. \tag{5.5}$$

If $v(\mathcal{A}) = 0$, then the limit is infinity.

Proof. Recall that $N_k^{(-\infty,x)}$ is the amount of time the walk spends at k before $T_{\geq x}$. Then for $x > c$,

$$\frac{T_{\geq x}}{x} = \frac{1}{x} \sum_{k=-\infty}^{c-1} N_k^{(-\infty,x)} + \frac{1}{x} \sum_{k=c}^{x-1} N_k^{(-\infty,x)}. \tag{5.6}$$

The first term approaches 0 almost surely by assumption (C4); hence, by Proposition 5.1.1 (2),

$$\lim_{x\to\infty} \frac{1}{x} \sum_{k=c}^{x-1} N_k^{(-\infty,x)} = \frac{1}{v(\mathcal{A})}, \mathbb{P}^a_{\mathcal{A}} - \text{a.s.} \tag{5.7}$$

We note that N_k and $N_k^{(-\infty,x)}$ differ only if the walk backtracks and visits k after reaching $[x, \infty)$. The sum, over all $k < x$, of these differences, is the total amount of time the walk spends to the left of x after $T_{\geq x}$, and it is bounded above by the time from $T_{\geq x}$ to the next regeneration time (defined as in (5.2)), which is in turn bounded above by $\tau_{J(x)} - \tau_{J(x)-1}$, where $J(x)$ is the (random) j such that $\tau_{j-1} \leq T_{\geq x} < \tau_j$. Hence

$$\frac{1}{x} \sum_{k=c}^{x-1} N_k^{(-\infty,x)} \leq \frac{1}{x} \sum_{k=c}^{x-1} N_k \leq \frac{1}{x} \sum_{k=c}^{x-1} N_k^{(-\infty,x)} + \frac{1}{x}[\tau_{J(x)} - \tau_{J(x)-1}] \tag{5.8}$$

Assume $v(\mathcal{A}) = 0$. Then by (5.7), the left side of (5.8) approaches ∞ as x approaches ∞, and therefore so does the middle.

On the other hand, suppose $v(\mathcal{A}) > 0$. By (5.1), $\mathbb{E}_{\mathcal{A}}[\tau_{M+2} - \tau_{M+1}] < \infty$. Then by Birkhoff's ergodic theorem, $\frac{\tau_n}{n} \to \mathbb{E}_{\mathcal{A}}[\tau_{M+2} - \tau_{M+1}] < \infty$, which implies that $\frac{\tau_n - \tau_{n-1}}{n}$ approaches 0. Since $J(x) \leq x + 1$, the term $\frac{1}{x}[\tau_{J(x)} - \tau_{J(x)-1}]$ approaches zero almost surely; hence the Squeeze Theorem yields the desired result. $\square$

Suppose for now that $R = 1$. Then, for almost every ω, it is possible to define a bi-infinite walk $\overline{\mathbf{X}} = (\overline{X}_t)_{t\in\mathbb{R}}$ whose "right halves" are distributed like random walks under ω with the acceleration function $\mathcal{A}$. From each site a, run a walk according to the transition probabilities and jump time probabilities given by ω and $\mathcal{A}(\omega)$ until it reaches $a + 1$ (which occurs in finite time $P^a_{\omega,\mathcal{A}}$–a.s. for P–a.e. ω). Concatenating all of these walks almost surely gives, up to a time shift[1], a unique walk $\overline{\mathbf{X}} = (\overline{X}_t)_{t\in\mathbb{R}}$ such that for any x, if it is known that $\overline{\mathbf{X}}$ jumps to x at time t, then $(\overline{X}_s)_{s\geq t}$ is distributed according to $P^x_{\omega,\mathcal{A}}$[2]. We may think of $\overline{\mathbf{X}}$ as a walk from $-\infty$ to ∞ in the environment ω with acceleration function $\mathcal{A}$.

In fact, with a bit more work, we can define a similar bi-infinite walk in the general case $R > 0$. Call the set of vertices $((k-1)R, kR]$ the *kth level* of $\mathbb{Z}$, and for $x \in \mathbb{Z}$, let $[[x]]_R$ denote the level containing x. Let ω be a given environment. From each point $a \in \mathbb{Z}$, run a walk according to the transition probabilities and jump time probabilities given by ω and $\mathcal{A}$ until it reaches the next level (i.e., $[[a + R]]_R$). This will happen $P^a_{\omega,\mathcal{A}}$–a.s. for P–a.e. ω, by transience to the right and because it is not possible to jump over a set of length R. Do this independently at every point for every level. This gives what we'll call a *cascade*: a set of (almost surely finite) walks indexed by $\mathbb{Z}$, where the walk indexed by $a \in \mathbb{Z}$ starts at a and ends upon reaching level $[[a + R]]_R$. Then for almost every cascade, concatenating these finite walks gives, for each point a, a right-infinite continuous-time walk $\mathbf{X}^a = (X^a_t)_{t\geq 0}$. Let $P_{\omega,\mathcal{A}}$ be the probability measure we have just described on the space of cascades, and let $E_{\omega,\mathcal{A}}$, $\mathbb{P}_{\mathcal{A}}$, and $\mathbb{E}_{\mathcal{A}}$ be correspondingly defined.

[1]↑Choose, for example, the time shift where $\overline{X}_0 = 0$ and where $\overline{X}_t < 0$ whenever $t < 0$.
[2]↑In order for the bi-infinite walk to be indexed by all real numbers, it must be true that the sum over all $x \geq 0$ of the time it takes for the walk started at x to hit $x + 1$ is infinite, and that the same sum over all $x < 0$ is also infinite. It is not hard to see that this is true with probability 1 for P–a.e. environment.

It is crucial to note that by the strong Markov property, the law of $\mathbf{X}^a$ under $P_{\omega,\mathcal{A}}$ is the same as the law of $\mathbf{X}$ under $P^a_{\omega,\mathcal{A}}$, which also implies that the law of $\mathbf{X}^a$ under $\mathbb{P}_{\mathcal{A}}$ is the same as the law of $\mathbf{X}$ under $\mathbb{P}^a_{\mathcal{A}}$.

For each $x \in \mathbb{Z}$, let the "coalescence event" C_x be the event that all the walks from level $[[x-R]]_R$ first hit level $[[x]]_R$ at x. On the event C_x, we say a coalescence occurs at x.

Lemma 5.1.4. *Let $\mathcal{E}_1$ be the event that all the $\mathbf{X}^a$ are transient to the right, that all steps to the left and right are bounded by L and R, respectively, and that infinitely many coalescences occur to the left and to the right of 0. Then $\mathbb{P}_{\mathcal{A}}(\mathcal{E}_1) = 1$.*

Proof. Boundedness of steps has probability 1 by assumption (C3), and by assumption (C4), all the walks $\mathbf{X}^a$ are transient to the right with probability 1. Now for $k \geq 2$ and $x \in \mathbb{Z}$, let $C_{x,k}$ be the event that all the walks from level $[[x-R]]_R$ first hit level $[[x]]_R$ at x without ever having reached $[[x-kR]]_R$. Choose k large enough that $\mathbb{P}_{\mathcal{A}}(C_{0,k}) > 0$; then and under the law $\mathbb{E}$, the events $\{C_{nkR,k}\}_{n\in\mathbb{Z}}$ are all independent and have equal, positive probability. Thus, infinitely many of them will occur in both directions, $\mathbb{P}_{\mathcal{A}}$–a.s. By definition, $C_{x,k} \subset C_x$, and so infinitely many of the events C_x occur in both directions, $\mathbb{P}_{\mathcal{A}}$–a.s. $\square$

Assume the environment and cascade are in the event $\mathcal{E}_1$. Let $(x_k)_{k\in\mathbb{Z}}$ be the locations of coalescence events (with x_0 the smallest non-negative x such that C_x occurs). By definition of the x_k, for every k and for every a to the left of $[[x_k]]_R$, $T_{[[x_k]]_R}(\mathbf{X}^a) = T_{x_k}(\mathbf{X}^a) < \infty$. Now for $j < k$, it necessarily holds that x_j is to the left of $[[x_k]]_R$, since there can be only one x_k per level. Define $\nu(j,k) := T_{x_k}(\mathbf{X}^{x_j})$. By definition of the walks $\mathbf{X}^a$, we have for $j < k$, $t \geq 0$,

$$X^{x_j}_{t+\nu(j,k)} = X^{x_k}_t. \tag{5.9}$$

From this one can easily check that the $\nu(j,k)$ are additive; that is, for $j < k < \ell$, we have $\nu(j,\ell) = \nu(j,k) + \nu(k,\ell)$. We note that for fixed j, $\nu(j,k)$ is increasing in k, because for $j < k < \ell$, the walk $\mathbf{X}^j$ must hit $[[x_k]]_R$ before it can hit $[[x_\ell]]_R$.

$$
\begin{aligned}
\nu(j,\ell) &= T_{x_\ell}(\mathbf{X}^{x_j}) && (5.10)\\
&= \inf\{t \geq 0 : X_t^{x_j} = x_\ell\} && (5.11)\\
&= \nu(j,k) + \inf\{t \geq 0 : X_{t+\nu(j,k)}^{x_j} = x_\ell\} && (5.12)\\
&= \nu(j,k) + \inf\{t \geq 0 : X_t^{x_k} = x_\ell\} && (5.13)\\
&= \nu(j,k) + T_{x_k}(\mathbf{X}^\ell) && (5.14)\\
&= \nu(j,k) + \nu(k,\ell) && (5.15)
\end{aligned}
$$

Moreover, with probability 1, $\lim_{j\to-\infty} \nu(j,0) = \infty$ and $\lim_{k\to\infty} \nu(0,k) = \infty$. This is because the bi-infinite walk visits each level at least once. For some $\varepsilon > 0$ and $\delta > 0$, there is positive probability $p > 0$ that for all i in a given level (e.g. $1 \leq i \leq R$), under the measure $P^i_{\omega,\mathcal{A}}$, the time to first jump is at least ε with probability $\geq \delta$. Becasue the $\mathcal{A}(\theta^i\omega)$ are stationary and ergodic under P, there are infinitely many levels in both directions where this is the case, and in each of those levels the bi-infinite walk independently spends at least ε units of time with $P_{\omega,\mathcal{A}}$-probability at least δ. Thus, there are $\mathbb{P}_{\mathcal{A}}$–a.s. infinitely many levels to the left and right of 0 where the bi-infinite walk spends at least ε units of time, and thus infinitely many j in both directions such that $\nu(j,j+1) \geq \varepsilon$. Let $\mathcal{E}_1'$ be the subset of $\mathcal{E}_1$ where this is also true, and assume the event $\mathcal{E}_1'$. Because all the $\mathbf{X}^{x_k}$ agree with each other in the sense of (5.9), we may define a single, bi-infinite walk $\overline{\mathbf{X}} = (\overline{X}_t)_{t\in\mathbb{R}}$ that agrees with all of the $\mathbf{X}^{x_k}$. For $t \geq 0$, let $\overline{X}_t = X_t^{x_0}$. For $t < 0$, choose $j < 0$ such that $\nu(j,0) > |t|$, and let $X_t = X^{x_j}_{\nu(j,0)-|t|}$. This definition is independent of the choice of j, because if $j < k < 0$, then by (5.9) and the additivity of the $\nu(j,k)$, we have

$$
X^{x_j}_{\nu(j,0)-|t|} = X^{x_j}_{\nu(j,k)+\nu(k,0)-|t|} = X^{x_k}_{\nu(k,0)-|t|}. \tag{5.16}
$$

We may then define $\overline{N}_x := \int_{-\infty}^{\infty} \mathbb{1}_{\overline{X}_t = x} dt$ to be the amount of time the walk $\overline{\mathbf{X}}$ spends at x. Thus, $\overline{N}_x = \lim_{a\to-\infty} N_x(\mathbf{X}^a)$.

Lemma 5.1.5. *Both of the sequences $(\mathbf{X}^a)_{a\in\mathbb{Z}}$ and $(\overline{N}_x)_{x\in\mathbb{Z}}$ are stationary and ergodic.*

Proof. For a given environment, the cascade that defines $\overline{\mathbf{X}}$ may be generated by a (countable) family $\mathbf{U} = (U_n^a)_{n\in\mathbb{N},a\in\mathbb{Z}}$ of i.i.d. uniform random variables on $[0,1]$. For such a collection, and an $a \in \mathbb{Z}$, let $\mathbf{U}^a$ be the projection $(U_n^a)_{n\in\mathbb{N}}$. Given an environment ω, the finite walk from a to level $[[a+R]]_R$ may be generated using the first several U_n^a. Let $\hat{\omega}^x = (\omega^x, \mathbf{U}^x)$, and $\hat{\omega} = (\hat{\omega}^x)_{x\in\mathbb{Z}}$. Define the left shift $\hat{\theta}$ by $\hat{\theta}(\hat{\omega}) := (\hat{\omega}^{x+1})_{x\in\mathbb{Z}}$. Then $(\hat{\omega}^x)_{x\in\mathbb{Z}}$ is an i.i.d. sequence. We have $\mathbf{X}^0 = \mathbf{X}^0(\hat{\omega})$ and $\mathbf{X}^a = \mathbf{X}^0(\hat{\theta}^a\hat{\omega})$. Similarly, $\overline{N}_0 = \overline{N}_0(\hat{\omega})$ and $N_x = \overline{N}_0(\hat{\theta}^x\hat{\omega})$. So it suffices to show that $\mathbf{X}^0$ and $\overline{N}_0$ are measurable.

The measurability of $\mathbf{X}^0$ is obvious. For $\overline{N}_0$, let $A_{k,s,B,r}$ be the event that:

(a) a coalescence event $C_{x,k}$ (as defined in the proof of Lemma 5.1.4) occurs with $-B \leq x - kR < x < 0$, so that $\overline{\mathbf{X}}$ agrees with $\mathbf{X}^x$ to the right of x;

(b) $N_0^{[-B,B]}(\mathbf{X}^x) \geq s$, where $N_0^{[-B,B]}$ is the amount of time the walk spends at x before exiting $[-B,B]$; and

(c) none of the walks from sites $a \in [-B,B]$ uses more than r of the random variables U_r^a.

This is the event that $\overline{N}_0$ is seen to be at least s by looking only within $[-B,B]$ and only at the first r uniform random variables at each site. The event $A_{k,s,B,r}$ is measurable, because it is a measurable function of finitely many random variables, and the event $\{\overline{N}_0 > s\}$ is simply the union over all r, then over all B, and then over all k of these events. Thus, $\overline{N}_0$ is measurable. $\square$

We now give the connection between $\overline{N}_0$ and the limiting velocity $v(\mathcal{A})$.

Lemma 5.1.6. $v(\mathcal{A}) = \frac{1}{\mathbb{E}_{\mathcal{A}}[\overline{N}_0]}$. *Consequently, the walk is ballistic if and only if* $\mathbb{E}_{\mathcal{A}}[\overline{N}_0] < \infty$.

We note that a similar formula for the limiting speed in the ballistic case can be obtained from [36, Theorem 6.12] for discrete-time RWRE on a strip, although the probabilistic interpretation is less explicit, and an ellipticity assumption that does not hold for Dirichlet RWRE is required.

Proof. By Birkhoff's Ergodic theorem, for any $c \in \mathbb{Z}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=c}^{n} \overline{N}_k = \mathbb{E}_{\mathcal{A}}[\overline{N}_0],\ \mathbb{P}_{\mathcal{A}} - \text{a.s.} \tag{5.17}$$

Fix $a \in \mathbb{Z}$. For large enough k, $N_k(\mathbf{X}^a) = \overline{N}_k$. We therefore get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=c}^{n} N_k(\mathbf{X}^a) = \mathbb{E}_{\mathcal{A}}[\overline{N}_0],\ \mathbb{P}_{\mathcal{A}} - \text{a.s.} \tag{5.18}$$

It follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=c}^{n} N_k(\mathbf{X}) = \mathbb{E}_{\mathcal{A}}[\overline{N}_0],\ \mathbb{P}^a_{\mathcal{A}} - \text{a.s.} \tag{5.19}$$

By Lemma 5.1.3, we get $v(\mathcal{A}) = \frac{1}{\mathbb{E}_{\mathcal{A}}[\overline{N}_0]}$. □

We know that $\lim_{x\to\infty} N_0(\mathbf{X}^{-x}) = \overline{N}_0$. We'd like take expectations on both sides and interchange the limit with expectations to get

$$\lim_{x\to\infty} \mathbb{E}_{\mathcal{A}}[N_0(\mathbf{X}^{-x})] = \mathbb{E}_{\mathcal{A}}[\overline{N}_0]. \tag{5.20}$$

We can do this under the assumption that $\mathbb{E}^0_{\mathcal{A}}[N_0]$ is finite.

Lemma 5.1.7. *Assume $\mathbb{E}^0_{\mathcal{A}}[N_0]$ is finite. Then*

$$\lim_{x\to\infty} \mathbb{E}_{\mathcal{A}}[N_0(\mathbf{X}^{-x})] = \mathbb{E}_{\mathcal{A}}[\overline{N}_0]. \tag{5.21}$$

Proof. Note that $\lim_{x\to\infty} N_0(\mathbf{X}^{-x}) = \overline{N}_0$, $\mathbb{P}_{\mathcal{A}}$–a.s. We wish to apply the dominated convergence theorem, and we note that $N_0(\mathbf{X}^{-x}) \le \max_{1-R\le z\le 0} N_0(\mathbf{X}^z)$ for all $x > R$. To see that the latter has finite expectation, we have

$$\begin{aligned}
\mathbb{E}_{\mathcal{A}}\left[\max_{1-R\le z\le 0} N_0(\mathbf{X}^z)\right] &\le \sum_{z=1-R}^{0} \mathbb{E}_{\mathcal{A}}\left[N_0(\mathbf{X}^z)\right] && (5.22)\\
&= \sum_{z=1-R}^{0} E\left[E_{\omega,\mathcal{A}}[N_0(\mathbf{X}^z)]\right] && (5.23)\\
&= \sum_{z=1-R}^{0} E\left[E^z_{\omega,\mathcal{A}}[N_0]\right] && (5.24)\\
&= \sum_{z=1-R}^{0} E\left[P^z_{\omega,\mathcal{A}}(T_0<\infty)E^0_{\omega,\mathcal{A}}[N_0]\right] && (5.25)\\
&\le \sum_{z=1-R}^{0} E\left[E^0_{\omega,\mathcal{A}}[N_0]\right] && (5.26)\\
&= R\mathbb{E}^0_{\mathcal{A}}[N_0] && (5.27)\\
&< \infty && (5.28)
\end{aligned}$$

Now, applying the dominated convergence theorem gives us

$$\begin{aligned}
\lim_{x\to\infty}\mathbb{E}^0_{\mathcal{A}}[N_x] &= \lim_{x\to\infty}\mathbb{E}^{-x}_{\mathcal{A}}[N_0] && (5.29)\\
&= \lim_{x\to\infty}\mathbb{E}_{\mathcal{A}}[N_0(\mathbf{X}^{-x})] && (5.30)\\
&= \mathbb{E}_{\mathcal{A}}\left[\lim_{x\to\infty} N_0(\mathbf{X}^{-x})\right] && (5.31)\\
&= \mathbb{E}_{\mathcal{A}}[\overline{N}_0] && (5.32)
\end{aligned}$$

$\square$

We now have all the components of Proposition 5.1.2.

Proof of Proposition 5.1.2. (1) We claim that $\mathbb{E}_{\mathcal{A}}[\overline{N}_0] \le \mathbb{E}^0_{\mathcal{A}}[N_0]$. If $\mathbb{E}^0_{\mathcal{A}}[N_0]=\infty$, this claim is trivial. Otherwise, apply Lemma 5.1.7 along with the fact that $\mathbb{E}^{-x}_{\mathcal{A}}[N_0] \le \mathbb{E}^0_{\mathcal{A}}[N_0]$.

Now if $\mathbb{E}^0_{\mathcal{A}}[N_0]<\infty$, it follows that $\mathbb{E}_{\mathcal{A}}[\overline{N}_0]<\infty$. By Lemma 5.1.6, this means $v(\mathcal{A})>0$.

(2) is just (1) applied to the discrete-time case (see Remark 2.4.1 on page 41). $\square$

5.2 Proof of main acceleration theorem

We now prove the main theorem of this chapter.

Theorem 5.2.1 (Theorem 2.4.1). *Assume $\kappa_1 \neq 0$. Then $P_{\mathcal{G}}$ has essential slowing if and only if $|\kappa_1| \leq 1$.*

Proof of Theorem 2.4.1. Assume without loss of generality that $\kappa_1 > 0$. We will first show that if $\kappa_1 > 1$, then there exists a Bouchet acceleration function $\mathcal{A}$ such that $v(\mathcal{A}) > 0$. Assume $\kappa_1 > 1$. Recall from Proposition 5.1.2 (1), that if $\mathbb{E}^0_{\mathcal{G},\mathcal{A}}[N_0] < \infty$, then $v(\mathcal{A}) > 0$, where $N_x = \int_0^\infty \mathbb{1}_{X_t=x} dt$ is the amount of time the walk spends at x. We will construct a Bouchet acceleration function $\mathcal{A}$ such that $v(\mathcal{A}) > 0$. Recall that for $x < y$, $N'_{x,y}$ is the number of backward trips from $[y, \infty)$ to $(-\infty, x]$. Choose M large enough that $\mathbb{E}^0_{\mathcal{G}}[N'_{-M,0} + N'_{0,M}] < \infty$ (This can be done by Proposition 4.2.5). Recall also that for a set $S \subseteq \mathbb{Z}$, and for $x \in S$, we have defined N^S_x to be the amount of time a walk spends at x before leaving S for the first time. For any environment ω, let $\mathcal{A}(\omega)$ be the distribution of any positive random variable with mean $\mu_{\mathcal{A}(\omega)} = \frac{1}{E^0_\omega\left[N_0^{[-M,M]}\right]}$. In this case, we have

$$
\begin{aligned}
E^0_{\omega,\mathcal{A}}[N_0] &= E^0_\omega[N_0]\ \mu_{\mathcal{A}(\omega)} && (5.33)\\
&= E^0_\omega\left[N_0^{[-M,M]}\right]\ E^0_\omega\left[1 + \#\begin{Bmatrix}\text{excursions}\\ \text{to}\quad [-M,M]^c\\ \text{and back to } 0\end{Bmatrix}\right]\ \mu_{\mathcal{A}(\omega)} && (5.34)\\
&= E^0_\omega\left[1 + \#\begin{Bmatrix}\text{excursions}\\ \text{to}\quad [-M,M]^c\\ \text{and back to } 0\end{Bmatrix}\right] && (5.35)\\
&\leq 1 + E^0_\omega[N'_{-M,0} + N'_{0,M}] && (5.36)
\end{aligned}
$$

Taking expectations with respect to $P_{\mathcal{G}}$ on both sides gives us $\mathbb{E}^0_{\mathcal{G},\mathcal{A}}[N_0] < \infty$. By Proposition 5.1.2 (1), $v(\mathcal{A}) > 0$ for this $\mathcal{A}$.

Next, we show that if $\kappa_1 \leq 1$, then $v(\mathcal{A}) = 0$ for any Bouchet acceleration function $\mathcal{A}$. Assume $\kappa_1 \leq 1$, and let $\mathcal{A}$ be a Bouchet acceleration function that depends only on the environment ω on $[-M, M]$. We wish to show that $v(\mathcal{A}) = 0$.

This part of the proof makes use of the constructions in Section 5.1. We let the "space of cascades" and the measures and random variables on that space be as defined there. By Lemma 5.1.6, $v(\mathcal{A}) = 0$ if $\mathbb{E}_{\mathcal{G},\mathcal{A}}\left[\overline{N}_0\right] = \infty$.

Now the distribution of the jump time at the site $-M-R$ is $\mathcal{A}(\theta^{-M-R}\omega)$, which depends only on $\omega^{\leq -R}$.

For $x < y$, let $\overline{N}_{x,y}$ be the number of trips from y to x that the walk $\overline{\mathbf{X}}$ takes. In other words, $\overline{N}_{x,y} = \lim_{a\to-\infty} N^a_{x,y}$. Now let $\overline{T}_{[1-R,0]}$ be the first time the walk $\overline{\mathbf{X}}$ hits $[1-R,0]$, and let $A = \overline{X}_{\overline{T}_{[1-R,0]}}$ be the location where $\overline{\mathbf{X}}$ first hits $[1-R,0]$. Then $\overline{N}_{-M-R,A} = N^A_{-M-R,A}$. We have

$$
\begin{aligned}
E_{\omega,\mathcal{A}}\left[\overline{N}_{-M-R}\right] &= E_{\omega}\left[\overline{N}_{-M-R}\right]\mu_{\mathcal{A}(\theta^{-M-R}\omega)} && (5.37)\\
&\geq E_{\omega}\left[\overline{N}_{-M-R,A}\right]\mu_{\mathcal{A}(\theta^{-M-R}\omega)} && (5.38)\\
&= E_{\omega}\left[N^A_{-M-R,A}\right]\mu_{\mathcal{A}(\theta^{-M-R}\omega)} && (5.39)\\
&\geq E_{\omega}\left[N^0_{-M-R,0}\mathbb{1}_{\{A=0\}}\right]\mu_{\mathcal{A}(\theta^{-M-R}\omega)} && (5.40)\\
&= E_{\omega}\left[N^0_{-M-R,0}\right]P_{\omega}(A=0)\mu_{\mathcal{A}(\theta^{-M-R}\omega)} && (5.41)\\
&= E^0_{\omega}\left[N_{-M-R,0}\right]P_{\omega}(A=0)\mu_{\mathcal{A}(\theta^{-M-R}\omega)}, && (5.42)
\end{aligned}
$$

where (5.41) comes from the fact that under P_ω, A depends only on the finite walks starting from y for $y \leq -R$, while $\mathbf{X}^0$ depends only on finite walks starting from sites $y \geq 0$. Now the terms $P_\omega(A=0)$ and $\mu_{\mathcal{A}(\theta^{-M-R}\omega)}$ in (5.42) are determined by $\omega^{\leq -R}$. Taking conditional expectations on both sides therefore gives us, with probability 1,

$$
E_{\mathcal{G}}\left[E_{\omega,\mathcal{A}}\left[\overline{N}_{-M-R}\right]\Big|\,\omega^{\leq -R}\right] \geq E_{\mathcal{G}}\left[E^0_{\omega}[N_{-M,0}]\Big|\omega^{\leq -R}\right]P_{\omega}(A=0)\mu_{\mathcal{A}(\theta^{-M-R}\omega)} \qquad (5.43)
$$

Claim 4.2.4.1 from the proof of Proposition 4.2.4 gives us

$$
E_{\mathcal{G}}\left[E^0_{\omega}[N_{-M-R,0}]\Big|\omega^{\leq -R}\right] = \infty,\ P_{\mathcal{G}} - a.s. \qquad (5.44)
$$

Hence the right hand side of (5.43) is infinite, $P_{\mathcal{G}}$–a.s. Taking expectation with respect to $P_{\mathcal{G}}$ in (5.43), we get $\mathbb{E}_{\mathcal{G},\mathcal{A}}\left[\overline{N}_{-M-R}\right] = \infty$. By stationarity of the $\{\overline{N}_x\}_{x\in\mathbb{Z}}$, it follows that $\mathbb{E}_{\mathcal{G},\mathcal{A}}\left[\overline{N}_0\right] = \infty$, and by Lemma 5.1.6, $v(\mathcal{A}) = 0$. □

6. OPEN QUESTIONS

The purpose of this chapter is to gather open questions and conjectures that are presented throughout this thesis.

6.1 Questions and conjectures about directional transience

The first conjecture says that a positive annealed probability of transience in one direction ℓ implies positive annealed probability of transience in all directions in a neighborhood of ℓ.

Conjecture (Conjecture 2.2.3)**.** *Let $\mathbb{P}^0$ be the law of an i.i.d. RWRE on $\mathbb{Z}^d$, and let S^{d-1} be the set of a unit vectors in $\mathbb{R}^d$. Then for all $\ell \in S^{d-1}$, if $\mathbb{P}^0(A_\ell) > 0$, then there exsts a neighborhood $U \subseteq S^{d-1}$ such that $\mathbb{P}^0(A_{\ell'}) > 0$ for all $\ell' \in U$.*

We defined A_ℓ^0 to be be the event that $\lim_{n\to\infty} X_n \cdot \ell = \infty$, but there is no neighborhood $U \in S^{d-1}$ containing ℓ such that for all $\ell' \in U$, $\lim_{n\to\infty} X_n \cdot \ell' = \infty$. This allows us to restate the above conjecture in a simple way.

Conjecture (Conjecture 2.2.3)**.** *Let $\mathbb{P}^0$ be the law of an i.i.d. RWRE on $\mathbb{Z}^d$. Then for all $\ell \in S^{d-1}$, $\mathbb{P}^0(A_\ell^0) = 0$.*

It also allows us to strengthen the conjecture.

Conjecture (Conjecture 3.4.1)**.** *Let $\mathbb{P}^0$ be the law of an i.i.d. RWRE on $\mathbb{Z}^d$. Then $\mathbb{P}^0\left(\bigcup_{\ell \in S^{d-1}} A_\ell^0\right) = 0$.*

We proved Conjecture 2.2.3 for RWDE with bounded jumps, but Conjecture 3.4.1 remains open even for nearest-neighbor RWDE on all $\mathbb{Z}^d$, $d \geq 2$.

6.2 Questions and conjectures about ballisticity

To prove Theorem 2.3.6, we showed first that $\kappa_0 \leq 1$ implies that the limiting speed is 0 due to finite trapping, and second that $\kappa_1 \leq 1$ implies that the limiting speed is 0 due to large-scale backtracking. It was then necessary to show that if neither effect on its own is strong enough to cause zero speed, then the two effects are independent enough that they cannot combine to cause zero speed, and thus that the walk is ballistic. Having shown this in the case of RWDE, we ask whether it is true for all RWRE on $\mathbb{Z}$ with bounded jumps. Recall Equation 2.1, which is true for every environment ω on $\mathbb{Z}$ and every $M > 0$.

$$E_\omega^0[N_0] = E_\omega^0\left[N_0^{[-M,M]}\right] E_\omega^0\left[\#\begin{Bmatrix}\text{Times exiting } [-M,M] \\ \text{and then returning to } 0\end{Bmatrix}\right]. \tag{2.1}$$

If each of the terms on the right has finite expectation, we ask whether the product has finite expectation.

Question (Question 2.3.1)**.** *Let P be a probability measure on $\Omega_\mathbb{Z}$ satisfying (C1), (C2), (C3), and (C4), under which both terms on the right of (2.1) have finite expectation for all M; that is, $E\left[E_\omega^0\left[N_0^{[-M,M]}\right]\right] < \infty$ and $E\left[E_\omega^0\left[\#\begin{Bmatrix}\textit{Times exiting } [-M,M] \\ \textit{and then returning to } 0\end{Bmatrix}\right]\right] < \infty$. Does it necessarily follow that $\mathbb{E}^0[N_0] = E[E_\omega^0[N_0]] < \infty$ (and thus that the walk is ballistic)?*

We now ask another question about Equation 2.1, which we have not asked in the body of the thesis.

Question 6.2.1. *Let P be a probability measure on $\Omega_\mathbb{Z}$ satisfying (C1), (C2), (C3), and (C4). If there is some M for which each term on the right side of Equation 2.1 has finite expectation under P, is it necessarily true that each of these terms has finite expectation for all M?*

A well known conjecture in d-dimensional RWRE states that under a uniform ellipticity assumption, directional transience implies ballisticity. Sabot and Tournier in [15] suggest a related conjecture that applies to all RWRE, uniformly elliptic or not. Our concept of "essential slowing" allows us to formally state this conjecture.

Conjecture (Conjecture 2.4.2)**.** *For irreducible, iid, directionally transient RWRE in $d \geq 2$, essential slowing is impossible. That is, there always exists a Bouchet acceleration function $\mathcal{A}$ such that $v(\mathcal{A}) \neq 0$.*

The concept of essential slowing also allows us to ask a question related to Question 2.4.1, but for d-dimensional RWRE.

Question (Question 2.4.1)**.** *Let P be a probability measure on $\Omega_{\mathbb{Z}^d}$ satisfying (C1), (C2), and (C3), nd suppose there is an almost-sure limiting direction. Suppose essential slowing does not occur, and also that* $E\left[E_\omega^0\left[\#\left\{\begin{matrix}\text{Times exiting } [-M,M]^d \\ \text{and then returning to } 0\end{matrix}\right\}\right]\right] < \infty$ *for all M. Does it necessarily follow that the walk is ballistic?*

In Proposition 4.2.1, we gave an algorithm to compute κ_0 given L, R, and the α_i. We also know that given the set $\mathcal{N}$, κ_0 is an elementary function (a minimum of finitely many positive integer combinations) of the α_i.

Question 6.2.2. *Find an algorithm to compute, given $\mathcal{N}$, the formula for κ_0 as an elementary function of the α_i.*

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A. AUXILIARY RESULTS

This appendix contains some proofs of minor results in order to save space in the body of the paper. The first is a weak 0-1 law.

Theorem (Kalikow's 0-1 Law)**.** *Let $\mathbb{P}^0$ be the annealed measure of a RWRE on $\mathbb{Z}^d$ satisfying assumptions (C1), (C2), and (C3). Then for every $\ell \in S^{d-1}$, $\mathbb{P}^0(A_\ell \cup A_{-\ell}) \in \{0,1\}$.*

A rudimentary version of the theorem for two dimensions, using a uniform ellipticity assumption and assuming $\ell = (0,1)$, was first given by Kalikow in [30]. Improvements were made in [37] (allowing general d and general ℓ) and [23] (removing the uniform ellipticity assumption), but the overall structure of the argument has changed very little. The proof in [23] does not use the nearest-neighbor assumption, except in a lemma that we now state and re-prove using the same ideas but without the nearest-neighbor assumption.

Lemma A.0.1. *Let $\mathbb{P}^0$ be the annealed measure of a RWRE on $\mathbb{Z}^d$ satisfying assumptions (C1), (C2), and (C3). Then for every $\ell \in S^{d-1}$ and $a < b \in \mathbb{R}$,*

$$\mathbb{P}^0(\#\{n \geq 0 : X_n \cdot \ell \geq a\} = \infty, T^\ell_{\geq b} = \infty) = 0. \tag{A.1}$$

Proof. Observe that on the event $\#\{n \geq 0 : X_n \cdot \ell \geq a\} = \infty$, it is either the case that for some y with $a \leq y \cdot \ell < b$, $X_n = y$ infinitely often, or that X_n hits infinitely many vertices in the slab $\{a \leq x \cdot \ell < b\}$. It therefore suffices to show that the intersection of each of these events with the event $T^\ell_{\geq b} = \infty$ has probability 0.

First, fix y with $a \leq y \cdot \ell < b$. By the irreducibility assumption (C2), $P^y_\omega(T^\ell_{\geq b} < \tilde{T}_y) > 0$ for almost every ω. For such an ω, the strong Markov property implies that the quenched probability of hitting y at least n times before $T^\ell_{\geq b}$ is no more than $P^y_\omega(\tilde{T}_x < T^\ell_{\geq b})^{n-1}$, which approaches 0 as $n \to \infty$. Thus, the (quenched or annealed) probability of hitting y infinitely many times without ever reaching the half-space $\{x \cdot \ell \geq b\}$ is 0. Summing over countably many y still gives a probability of 0.

Now consider the event that infinitely many points in $\{a \leq x \cdot \ell < b\}$ are hit. By assumption (C2), each of these points x has a *possible path* (in the notation introduced in the proof of Theorem 2.1.1) to $\{x \cdot \ell \geq b\}$. By shift-invariance, there is some $N > 0$ and $\varepsilon > 0$ such that each x in $\{a \leq x \cdot \ell < b\}$ has a possible path of length no more than N and with annealed probability at least ε. Thus, in order to hit infinitely many points in $\{a \leq x \cdot \ell < b\}$, the walk must hit the vertex sets of infinitely many disjoint paths to $\{x \cdot \ell \geq b\}$, each of which has length no more than N and annealed probability at least ε. Now by the i.i.d. assumption (C1), each time the walk hits an unexplored vertex set of such a path, its probability, conditioned on its entire past, of immediately taking (the rest of) that path is at least ε. The probability of hitting vertex sets of n unique such paths before hitting $\{x \cdot \ell \geq b\}$ is therefore no more than $(1-\varepsilon)^{n-1}$, which approaches 0 as $n \to \infty$. Thus, the annealed probability of hitting the vertex sets of infinitely many disjoint paths to $\{x \cdot \ell \geq b\}$, each with length no more than N and annealed probability at least ε, without ever reaching $\{x \cdot \ell \geq b\}$, is 0. Since these paths have no more than N vertices in them, hitting infinitely many sites in $\{a \leq x \cdot \ell < b\}$ requires hitting the disjoint vertex sets of infinitely many such paths, and therefore the annealed probability of hitting infinitely many sites in $\{a \leq x \cdot \ell < b\}$ without ever reaching $\{x \cdot \ell \geq b\}$ is 0. This gives us (A.1). ☐

We outline a proof of Proposition 4.1.1.

Recall that we have defined $\tau_0 := 0$, and for $k \geq 1$, $\tau_k := \min\{n > \tau_{k-1} : X_n > X_j \text{ for all } j < n, X_n \leq X_j \text{ for all } j > n\}$. The proposition we are to prove is the following.

Proposition (Proposition 4.1.1)**.** *Let P be a probability measure on $\Omega_{\mathbb{Z}}$ satisfying (C1), (C2), and (C3). Then the following hold:*

1. *There is a $\mathbb{P}^0$–almost sure limiting velocity*

$$v := \lim_{n\to\infty} \frac{X_n}{n} = \frac{\mathbb{E}^0[X_{\tau_2} - X_{\tau_1}]}{\mathbb{E}^0[\tau_2 - \tau_1]}, \tag{A.2}$$

   where the numerator is always finite, and the fraction is understood to be 0 if the denominator is infinite.

2. $\lim_{x\to\infty} \frac{T_{\geq x}}{x} = \frac{1}{v}$*, where* $\frac{1}{v}$ *is understood to be* ∞ *if* $v = 0$*.*

As we mentioned after the statement of the proposition, the recurrent case is handled by [34] (with slight modifications). Therefore, we may assume the walk is transient to the right almost surely. It is standard (see, for example, [35], [37]) to prove a LLN like (A.2) under an assumption of directional transience by the following steps:

(a) Show that $\frac{X_{\tau_k}}{k}$ approaches $\mathbb{E}^0[X_{\tau_2} - X_{\tau_1}]$

(b) Show that $\frac{\tau_k}{k}$ approaches $\mathbb{E}^0[\tau_2 - \tau_1]$

(c) Show that $\mathbb{E}^0[X_{\tau_2} - X_{\tau_1}] < \infty$

(d) Conclude that the limit (A.2) holds for the subsequence $\left(\frac{X_{\tau_k}}{\tau_k}\right)_k$

(e) Use straightforward bounds that come from the definitions of the τ_k to get the limit for the entire sequence $\left(\frac{X_n}{n}\right)_n$.

Part (2) then follows from a comparison of $\frac{x}{T_{\geq x}}$ with a subsequence of $\frac{X_n}{n}$.

The definition of the regeneration times is precisely set up so that both the sequences $(\tau_k - \tau_{k-1})_{k\geq 2}$ and $(X_{\tau_k} - X_{\tau_{k-1}})_{k\geq 2}$ are i.i.d. sequences, so proving the limits (a) and (b) is a matter of tracing how the i.i.d. feature follows from the definitions and applying the strong law of large numbers. In fact, arguing as in [35, Lemma 1], one can show that the triples

$$\xi_n := \left(\tau_n - \tau_{n-1} \ , \ (X_{\tau_{n-1}+i} - X_{\tau_{n-1}})_{i=1}^{\tau_n - \tau_{n-1}} \ , \ (\omega^x)_{x=X_{\tau_{n-1}}}^{X_{\tau_n}-1}\right)$$

are i.i.d. under $\mathbb{P}^0 = P \times P_\omega^0$ for $n \geq 2$.

We show the finiteness in (c) using arguments along the lines of those in [38, Lemma 3.2.5].[1] For $z \geq 0$, let B_z be the event that for some k, $X_{\tau_k} \in [zR, (z+1)R)$. Then

$$\begin{aligned}
\mathbb{P}^0(B_z) &= E\left[P^0_\omega(B_z)\right] \\
&\geq E\left[\sum_{i=0}^{R-1} P^0_\omega(T_{[zR,(z+1)R)} = zR+i) P^{zR+i}_\omega(T_{<zR+i} = \infty)\right] \\
&= \sum_{i=0}^{R-1} E\left[P^0_\omega(T_{[zR,(z+1)R)} = zR+i) P^{zR+i}_\omega(T_{<zR+i} = \infty)\right] \\
&= \sum_{i=0}^{R-1} \mathbb{P}^0(T_{[zR,(z+1)R)} = zR+i) \mathbb{P}^{zR+i}(T_{<zR+i} = \infty) \\
&= \mathbb{P}^0(T_{<0} = \infty),
\end{aligned} \tag{A.3}$$

where the second to last equality comes from the fact that $\omega^{<zR}$ is independent of $\omega^{\geq zR+i}$, and the last comes from translation invariance and the fact that $T_{[zR,(z+1)R)} < \infty$ $\mathbb{P}^0$–a.s. On the other hand, let B^2_z be the event that for some $k \geq 2$, $X_{\tau_k} \in [zR, (z+1)R)$. Then since $\lim_{z\to\infty} \mathbb{P}^0(X_{\tau_1} \geq zR) = 0$, we have

$$\begin{aligned}
\liminf_{z\to\infty} \mathbb{P}^0(B_z) &= \liminf_{z\to\infty} \mathbb{P}^0(B^2_z). \\
&= \liminf_{z\to\infty} \sum_{y\geq 1} \mathbb{P}^0\left(B^2_z, X_{\tau_1} = y\right) \\
&= \liminf_{z\to\infty} \sum_{y\geq 1} \mathbb{P}^0\left(\exists k \geq 2 : X_{\tau_k} - X_{\tau_1} \in [zR - y, (z+1)R - y), X_{\tau_1} = y\right) \\
&= \liminf_{z\to\infty} \sum_{y\geq 1} \mathbb{P}^0\left(X_{\tau_1} = y\right) \mathbb{P}^0\left(\exists k \geq 2 : X_{\tau_k} - X_{\tau_1} \in [zR - y, (z+1)R - y)\right) \\
&\geq \liminf_{z\to\infty} \sum_{y\geq 1} \mathbb{P}^0\left(X_{\tau_1} = y\right) \mathbb{P}^0\left(\exists k \geq 2 : X_{\tau_k} - X_{\tau_1} = zR\right).
\end{aligned}$$

But recall that by the renewal theorem (and by our irreducibility assumption (C2)),

$$\lim_{z\to\infty} \mathbb{P}^0\left(\exists k \geq 2 : X_{\tau_k} - X_{\tau_1} = zR\right) = \frac{1}{\mathbb{E}^0\left[X_{\tau_2} - X_{\tau_1}\right]},$$

[1] ↑Because we have assumed almost-sure transience to the right, the measure $\mathbb{Q}$ introduced there is unnecessary. Another difference is that in our model, transience to the right does not imply that every vertex to the right of the origin is hit. So instead of studying the probability, for a given x, that a regeneration occurs at site x, we focus on the probability that the regeneration occurs on a given interval of length R.

and hence, by dominated convergence,

$$\lim_{z\to\infty} \mathbb{P}^0(B_z) = \frac{\sum_{y\geq 1} \mathbb{P}^0\left(X_{\tau_1} = y\right)}{\mathbb{E}^0\left[X_{\tau_2} - X_{\tau_1}\right]} = \frac{1}{\mathbb{E}^0\left[X_{\tau_2} - X_{\tau_1}\right]}. \tag{A.4}$$

Comparing (A.3) and (A.4), we conclude that

$$\mathbb{E}^0\left[X_{\tau_2} - X_{\tau_1}\right] \leq \frac{1}{\mathbb{P}^0(T_{<0} = \infty)} < \infty.$$

This gives us the finiteness in (c), and (d) and (e) easily follow.

We now prove a lemma that is used to justify arguments where we replace transition probabilities at a particular site x with an almost-sure jump to another site y.

Lemma A.0.2. *Consider an environment ω on a finite or countable vertex set V. Assume the Markov chain is irreducible under ω. Let A and B be disjoint finite subsets of V, with A nonempty. Now let $x \in V - A \cup B$ and $y \in V - B$, with $P_\omega^y(T_A < T_B) \geq P_\omega^x(T_A < T_B)$. Suppose ω' is an environment that agrees with ω at all sites other than x but has $\omega'(x, y) = 1$. Then for all $z \in V$, $P_{\omega'}^z(T_A < T_B) \geq P_\omega^z(T_A < T_B)$*

Proof. We first assume that V is finite. For $z \in V$, let $f_0(z) = P_\omega^z(T_A < T_B)$. Then, for $i \geq 1$, define

$$f_i(z) := \begin{cases} \sum_{w\in V} \omega'(z, w) f_{i-1}(w) & z \notin A \cup B \\ 0 & z \in B \\ 1 & z \in A \end{cases}.$$

Now for $z \neq x$, $f_1(z) = f_0(z)$, since f is harmonic with respect to ω. On the other hand, $f_1(x) = f_0(y) \geq f_0(x)$. Thus, $f_1(z) \geq f_0(z)$ for all z. A straightforward induction now shows that for all $z \in V$, the sequence $(f_i(z))$ is increasing and bounded. Hence $f(z) := \lim_{i\to\infty} f_i(z)$ exists, and $f(z) \geq f_0(z)$ for all z.

We will have completed the proof if we can show that $f(z) = P_{\omega'}^z(T_A < T_B)$. One can easily check that f is ω'-harmonic on $V - A \cup B$, identically 1 on A, and identically 0 on B. The function $z \to P_{\omega'}^z(T_A < T_B)$ has these same properties, and by the maximum principle there is only one such function.

Now suppose V is infinite, and let $z_0 \in V$. Take any finite subset $S \subset V$ containing A, B, x, y, and z_0, and take one "sink" vertex ∂ not in S. For $z, w \in S$, define $\omega_*(z, w) := P_\omega^z(\tilde{T}_S < \infty, X_{\tilde{T}_S} = w)$, and $\omega_*(z, \partial) := P_\omega^z(\tilde{T}_S = \infty)$. Define $(\omega')_*$ similarly. Then for all $z \in S$,

$$P_\omega^z(T_A < T_B) = P_{\omega_*}^z(T_A < T_B). \tag{A.5}$$

To see this, it is straightforward to check that $z \to P_\omega^z(T_A < T_B)$ is harmonic with respect to ω_* on $S - \{A \cup B\}$, identically 1 on A, and identically 0 on $B \cup \{\partial\}$. The function $z \to P_{\omega_*}^z(T_A < T_B)$ has these same properties, and there is only one such function. Similarly,

$$P_{\omega'}^z(T_A < T_B) = P_{(\omega')_*}^z(T_A < T_B). \tag{A.6}$$

We want to show that $(\omega')_*$ has the same properties relative to ω_* that ω' has relative to ω, so that we may apply the conclusion from the finite case with $(\omega_*)' := (\omega')_*$. That is, we need to show that

(a) For $z \neq x$, and $w \in S \cup \{\partial\}$, we have $(\omega')_*(z, w) = \omega_*(z, w)$.

(b) $(\omega')_*(x, y) = 1$.

The statement (b) comes from the fact that $\omega'(x, y) = 1$. The statement (a) says that $P_{\omega'}^z(\tilde{T}_S < \infty, X_{\tilde{T}_S} = w) = P_\omega^z(\tilde{T}_S < \infty, X_{\tilde{T}_S} = w)$. This is true because these probabilities do not depend on the environment at x, and $\omega \equiv \omega'$ everywhere else.

We may now apply the finite case to the measures ω_* and $(\omega')_*$, concluding that $P_{(\omega')_*}^{z_0}(T_A < T_B) \geq P_{\omega_*}^{z_0}(T_A < T_B)$. By (A.5) and (A.6), we get $P_{\omega'}^{z_0}(T_A < T_B) \geq P_\omega^{z_0}(T_A < T_B)$. Since z_0 was arbitrary, this is enough to conclude the argument. $\square$

B. CALCULATING κ_0

In the body of the paper, we gave an algorithm to compute κ_0, but this algorithm grows in complexity with the smallest positive α_i. Here, we prove Proposition 4.2.2, which asserts that given an underlying directed graph, κ_0 is a minimum of finitely many positive integer combinations of the α_i. Although we do not have a general algorithm to find the formula, we give several examples where we are able to do so. These examples exhibit various important features that sets S attaining $\beta_S = \kappa_0$ may have. We also prove Proposition B.0.2, showing that κ_0 and κ_1 are unrestricted by each other. We begin with the following lemma.

Lemma B.0.1. *Let $\mathcal{T} \subseteq \mathbb{N}^k$ be a set of ordered k-tuples of positive integers. Let $\preceq$ be the natural partial ordering on $\mathbb{N}^k$, $(n_1, \ldots, n_k) \preceq (n_1', \ldots, n_k')$ if $n_i \leq n_i'$ for all i. Then there is a finite subset $\mathcal{T}^* \subseteq \mathcal{T}$ such that for all $\mathbf{x} \in \mathcal{T}$, there is an $\mathbf{x}^* \in \mathcal{T}^*$ such that $\mathbf{x}^* \preceq \mathbf{x}$.*

Proof. We prove this by induction on k. The base case $k = 1$ is trivial; a 1-tuple is simply a positive integer, and we can let $\mathcal{T}^* := \{\min \mathcal{T}\}$. Now suppose the result is true for all subsets of $\mathbb{N}^k$, and let $\mathcal{T} \subseteq \mathbb{N}^{k+1}$. Now let $\underline{\mathcal{T}} := \{(n_1, \ldots, n_k) : (n_1, \ldots, n_k, n_{k+1} \in \mathcal{T}$ for some $n_{k+1} \in \mathbb{N}\}$ be the projection of $\mathcal{T}$ onto $\mathbb{N}^k$, and for $n \in \mathbb{N}$ define $\underline{\mathcal{T}}(n) := \{(n_1, \ldots, n_k) : (n_1, \ldots, n_k, n) \in \mathcal{T}\}$. Thus, $\underline{\mathcal{T}} = \bigcup_{n=1}^{\infty} \underline{\mathcal{T}}(n)$. Now by the inductive hypothesis, there is a finite set $\underline{\mathcal{T}}^* \subseteq \underline{\mathcal{T}}$ such that every element of $\underline{\mathcal{T}}$ is greater than some element of $\underline{\mathcal{T}}^*$. Since $\underline{\mathcal{T}}^*$ is finite, $\underline{\mathcal{T}}^* \subseteq \bigcup_{n=1}^{N} \underline{\mathcal{T}}(n)$ for some N. Applying the inductive hypothesis to each $\underline{\mathcal{T}}(n)$ gives us sets $\underline{\mathcal{T}}^*(n)$ such that for all $\mathbf{x} \in \underline{\mathcal{T}}(n)$, there is a $\mathbf{x}^* \in \underline{\mathcal{T}}^*(n)$ such that $\mathbf{x}^* \preceq \mathbf{x}$.

Now define $\mathcal{T}^* := \bigcup_{n=1}^{N} \{(n_1, \ldots, n_k, n) : (n_1, \ldots, n_k) \in \underline{\mathcal{T}}^*(n)\}$. It is easy to see that this is a finite subset of $\mathcal{T}$. Now suppose $(n_1, \ldots, n_k, n_{k+1}) \in \mathcal{T}$. Suppose $n_{k+1} < N$. Then $(n_1, \ldots, n_k) \in \underline{\mathcal{T}}(n_{k+1})$, so there exists $(n_1^*, \ldots, n_k^*) \in \underline{\mathcal{T}}^*(n_{k+1})$ with $n_i^* \leq n_i$ for $1 \leq i \leq k$. Since $(n_1^*, \ldots, n_k^*, n_{k+1}) \in \mathcal{T}^*$, we are done.

On the other hand, suppose $n_{k+1} \geq N$. Then since $(n_1, \ldots, n_k) \in \underline{\mathcal{T}}$, there exists $(n_1^*, \ldots, n_k^*) \in \underline{\mathcal{T}}^*$ such that $n_i^* \leq n_i$ for all $i = 1, \ldots, k$. Now $(n_1^*, \ldots, n_k^*) \in \underline{\mathcal{T}}(n)$ for some $n \leq N \leq n_{k+1}$. Hence there exists $(n_1^{**}, \ldots, n_k^{**}) \in \underline{\mathcal{T}}^*(n)$ such that $n_i^{**} \leq n_i^*$ for all $1 \leq i \leq k$. Thus $(n_1^{**}, \ldots, n_k^{**}, n) \in \mathcal{T}^*$ with $n_i^{**} \leq n_i$ for $1 \leq i \leq k$, and $n \leq n_{k+1}$. $\square$

We are now able to prove Proposition 4.2.2.

Proof of Proposition 4.2.2. For any finite set $S \subset \mathbb{Z}$, β_S is a sum of weights of edges exiting β. The weight of each edge is α_i for some i, and each α_i must be included at least once, as the weight of an edge exiting either the rightmost or leftmost point of S. Thus,

$$\beta_S = \sum_{i=-L}^{R} x_i \alpha_i$$

where $x_i = x_i(S) := \#\{z \in S : z + i \notin S\} \geq 1$.

Now let $\mathcal{T} \subset \mathbb{N}^{R+L}$ be the set of ordered tuples $(y_{-L}, \ldots, y_R)$ such that there is some finite set S with $x_i(S) = y_i$ for all $-L \leq i \leq R$. Thus,

$$\kappa_0 = \inf \left\{ \sum_{i=-L}^{R} y_i \alpha_i : (y_{-L}, \ldots, y_R) \in \mathcal{T} \right\}.$$

Applying Lemma B.0.1, we get a finite set $\mathcal{T}^* \subseteq \mathcal{T}$ such that for any S, there is a $(y_{-L}, \ldots, y_R) \in \mathcal{T}^*$ with $y_i \leq x_i(S)$ for all $-L \leq i \leq R$. Thus,

$$\kappa_0 = \min \left\{ \sum_{i=-L}^{R} y_i \alpha_i : (y_{-L}, \ldots, y_R) \in \mathcal{T}^* \right\}. \tag{B.1}$$

This is a minimum of finitely many positive integer combinations of the α_i. $\square$

We now give examples where we can find the formula for κ_0. Recall that $\kappa_0 := \inf\{\beta_S : S \subset \mathbb{Z}$ finite, strongly connected$\}$, where β_S is the sum of edge weights leaving the set S. By shift invariance of the graph $\mathcal{G}$, it suffices to consider sets S whose leftmost point is 0.

We already showed in Claim 4.2.1.1 that $\kappa_0 \leq d_+ + d^-$. We can also give a general lower bound: $\kappa_0 \geq c^+ + c^-$. This is because any strongly connected set will have weight at least c^+ exiting from the rightmost point and weight at least c^- exiting from the leftmost point. Therefore, every strongly connected set S has $\beta_S \geq c^+ + c^-$, and taking the infimum preserves the inequality. So we have the bounds

$$c^+ + c^- \leq \kappa_0 \leq d^+ + d^-. \tag{B.2}$$

Example B.0.1. $L = R = 1$.

In this case, $d^+ = c^+ = \alpha_1$, and $d^- = c^- = \alpha_{-1}$, so (B.2) immediately implies $\kappa_0 = \alpha_1 + \alpha_{-1}$. This can also be seen by noting that the only strongly connected sets are intervals, which all have the same exit weight.

Example B.0.2. $\alpha_0 > 0$.

In this case, $\{0\}$ is already a strongly connected set, so (B.2) gives $\kappa_0 = \beta_{\{0\}} = c^+ + c^-$.

Example B.0.3. $L = 2$, $R = 3$, $\alpha_i = 0$ for $i = -1, \ldots, 2$.

In this case, we also have $\kappa_0 = d^+ + d^-$. Let S be a strongly connected finite set of vertices with left endpoint 0. Then S contains 3, since there must be a vertex reachable from 0 in one step, and by assumption there are no vertices to the left of 0. Also, S contains 2, since 0 must be reachable in one step from a vertex in S. Likewise, 2 must then also be reachable, and since $-1 \notin S$, S must contain 4 as well. Now since a vertex must be reachable from 3, S must contain either 1 or 6. Suppose S contains 1. Then S contains $[0, 4]$, which has exit weight $d^+ + d^-$, and by Claim 4.2.1.2, $\beta_S \geq \beta_{[0,4]} = d^+ + d^-$. On the other hand, suppose S does not contain 1. Then it contains 6. If S also contains 5, then S contains the interval $[2, 6]$, which is shift-equivalent to $[0, 4]$. If S contains neither 1 nor 5, then it contains exactly the set $\{0, 2, 3, 4, 6\}$ and possibly vertices to the right and/or left of this set, so by Claim 4.2.1.2, $\beta_S \geq \beta_{\{0,2,3,4,6\}}$. One can easily check that in this case, $\beta_{\{0,2,3,4,6\}} = d^+ + d^-$.

We have calculated κ_0 without even showing that either $[0, 4]$ or $\{0, 2, 3, 4, 6\}$ is strongly connected, but in fact they both are. Consider the path $0 \to 3 \to 1 \to 4 \to 2 \to 0$ in $[0, 4]$ and the path $0 \to 3 \to 6 \to 4 \to 2 \to 0$ in $\{0, 2, 3, 4, 6\}$. Thus, even when $\kappa_0 = d^+ + d^-$, a minimizing set S for β_S need not be an interval (although a large enough interval will always be a minimizing set).

Example B.0.4. $L = 1$, $R \geq 2$, $\alpha_0 = 0$ $\alpha_1 > 0$, $\alpha_i = 0$ for $i = 2, \ldots, R - 1$.

In this case, we show that $\kappa_0 = 2\alpha_R + \alpha_1 + \alpha_{-1}$; thus if $R > 2$, then $\kappa_0 < d^+ + d^-$. Let S be a strongly connected set with left endpoint 0. Since 0 must be reachable from another point in S, we have $1 \in S$. Now by Claim 4.2.1.2, this implies $\beta_S \geq \beta_{\{0,1\}} = 2\alpha_R + \alpha_1 + \alpha_{-1}$. The set $\{0, 1\}$ is strongly connected, and hence $\kappa_0 = \beta_{\{0,1\}} = 2\alpha_R + \alpha_1 + \alpha_{-1}$.

This example lets us show that κ_0 and κ_1 are independent in the sense that no information about either may be inferred from the other.

Proposition B.0.2. *The ordered pair* (κ_0, κ_1) *may take on any value in the first quadrant of* $\mathbb{R}^2$.

Proof. Let $a, b > 0$. We will show that L, R, and the α_i may be chosen such that $\kappa_0 = a$ and $\kappa_0 = b$. Let $L = 1$, and let $R \geq 2$ be large enough that $\frac{a}{2} > \frac{b}{R}$. Then let $\alpha_1 = \alpha_{-1} = \frac{a}{2} - \frac{b}{R}$, $\alpha_R = \frac{b}{R}$, and all other $\alpha_i = 0$. Then $\kappa_1 = -\alpha_{-1} + \alpha_1 + R\alpha_R = b$, and by Example B.0.4, $\kappa_0 = a$. □

Example B.0.5. $L = R = 2$, $\alpha_{-1}, \alpha_1 > 0$, $\alpha_0 = 0$.

There are two possibilities. Let S be a strongly connected set with leftmost point 0. If $1 \in S$, then by Claim 4.2.1.2, $\beta_S \geq \beta_{\{0,1\}} = d^+ + d^-$. On the other hand, if $1 \notin S$, then $2 \in S$, and by Claim 4.2.1.2, $\beta_S \geq \beta_{\{0,2\}} = \alpha_{-2} + 2\alpha_{-1} + 2\alpha_1 + \alpha_2$. Since both $\{0, 1\}$ and $\{0, 2\}$ are strongly connected, κ_0 may be either $d^+ + d^-$ or $\alpha_{-2} + 2\alpha_{-1} + 2\alpha_1 + \alpha_2$, depending on whether $\alpha_{-1} + \alpha_1$ or $\alpha_{-2} + \alpha_2$ is smaller. That is,

$$\begin{aligned}\kappa_0 &= \min(2\alpha_{-2} + \alpha_{-1} + \alpha_1 + \alpha_2, \alpha_{-2} + 2\alpha_{-1} + 2\alpha_1 + \alpha_2)\\ &= \alpha_{-2} + \alpha_{-1} + \alpha_1 + \alpha_2 + \min(\alpha_{-1} + \alpha_1, \alpha_{-2} + \alpha_2)\\ &= \min(\beta_{\{0,1\}}, \beta_{\{0,2\}}).\end{aligned}$$

Example B.0.6. $L = 6$, $R = 3$, $\alpha_2 > 0$, $\alpha_i = 0$ for $i = -5, \ldots, 1$.

If S is a finite, strongly connected set with 0 the leftmost vertex, then $6 \in S$, since 0 must be reachable from the right. We consider possible sets $S \cap [0, 6]$. There are 32 subsets of $[0, 6]$ that contain 0 and 6; however, S must contain either 2 or 3, since there must be edges from 0 to other sets in S and nothing to the left of 0 is allowed. Similarly, if S contains 1, then it must contain either 3 or 4, and if S contains 2, then it must contain either 4 or 5. This eliminates 12 of the 32 possibilities, leaving 20 possibilities for $S \cap [0, 6]$. Of these, we first consider two candidates, $\{0, 3, 6\}$ and $\{0, 2, 4, 6\}$. Both of these are strongly connected, and so $\beta_{\{0,3,6\}} = 2\alpha_{-6} + 3\alpha_2 + \alpha_3$ and $\beta_{\{0,2,4,6\}} = 3\alpha_{-6} + \alpha_2 + 4\alpha_3$ both provide upper bounds for κ_0. Depending on the values of the α_i, either can be lower than the other. The set $\{0, 2, 3, 4, 6\}$ is also strongly connected, but has $\beta_{\{0,2,3,4,6\}} = 4\alpha_{-6} + 2\alpha_2 + 3\alpha_3$. Thus, if $\alpha_2 \geq \alpha_3$, then $\beta_{\{0,2,4,6\}} < \beta_{\{0,2,3,4,6\}}$, and if $\alpha_2 \leq \alpha_3$, then $\beta_{\{0,3,6\}} < \beta_{\{0,2,3,4,6\}}$. One can simply check that other 17 of the possible sets $D = S \cap [0,6]$ either have $\beta_D > \beta_{\{0,3,6\}}$ for all possible values of the α_i, $\beta_D > \beta_{\{0,2,4,6\}}$ for all possible values of the α_i, or $\beta_D > \beta_{\{0,2,3,4,6\}}$ for all possible values of the α_i. By Claim 4.2.1.2, this implies that $\beta_S \geq \min(\beta_{\{0,3,6\}}, \beta_{\{0,2,4,6\}})$. Therefore,

$$\begin{aligned}\kappa_0 &= \min(2\alpha_{-6} + 3\alpha_2 + \alpha_3, 3\alpha_{-6} + \alpha_2 + 4\alpha_3) \\ &= \min(\beta_{\{0,3,6\}}, \beta_{\{0,2,4,6\}}).\end{aligned}$$

In all five of the above examples, there is always a set S minimizing β_S that represents a single, simple loop. The exit time from S is the first time the walk stops repeating this loop. Thus, if $\kappa_0 \leq 1$, then there is a single loop that the walk is expected to repeat infinitely many times before deviating from it.

In the nearest-neighbor case, treated in Example B.0.1, $\kappa_0 \leq 1$ means the walk is expected to repeat the loop $0 \to 1 \to 0$ infinitely many times before ever taking a different step (and, likewise, the walk is expected to repeat the loop $0 \to -1 \to 0$ infinitely many times before ever stepping to 1). This does not mean the only finite traps are sets of the form $\{x, x+1\}$. For example, it is also the case that $\beta_{[0,5]} \leq 1$, so that the walk is expected to spend an infinite amount of time in $[0,5]$ before leaving it, regardless of the precise path (and even if transition probabilities at sites 1,2,3, and 4 are conditioned to be moderate). But there are no finite traps "worse" (in the sense of finite moments of quenched expected exit time) than the set $\{0,1\}$.

In fact, for nearest-neighbor RWDE on $\mathbb{Z}^d$, pairs of adjacent vertices are always the worst finite traps, and if $\kappa_0 \leq 1$, then the walk is expected to bounce back and forth between 0 and one other vertex infinitely many times before doing anything else [15].

Our other examples so far match this trend in a sense; although the worst traps are not necessarily pairs of vertices, the worst traps are loops, and $\kappa_0 \leq 1$ means there is a loop that the walk is expected to iterate infinitely many times before doing anything else.

- In Example B.0.2, one such loop is $0 \to 0$.
- In Example B.0.3, one such loop is $0 \to 3 \to 6 \to 4 \to 2 \to 0$.
- In Example B.0.4, one such loop is $0 \to 1 \to 0$.

- In Example B.0.5, one such loop is $0 \to 1 \to 0$ (if $\beta_{\{0,1\}} \le 1$) or $0 \to 2 \to 0$ (if $\beta_{\{0,2\}} \le 1$).
- In Example B.0.6, one such loop is $0 \to 3 \to 6 \to 0$ (if $\beta_{\{0,3,6\}} < 1$) or $0 \to 2 \to 4 \to 6$ (if $\beta_{\{0,2,4,6\}} < 1$).

Our next example shows that unlike in the nearest-neighbor case on $\mathbb{Z}^d$, there are parameters where the strongest finite traps never represent just one loop. In particular, one can find cases where $\kappa_0 \le 1$, so there are finite traps in which the walk is expected to be stuck for an infinite amount of time, but there is no single loop that the walk is expected to iterate infinitely many times before deviating from it.

Example B.0.7. $L = R = 2$, $\alpha_{-1} = \alpha_0 = 0$, $\alpha_1 > 0$.

A finite, strongly connected set with 0 as its leftmost point will necessarily contain 2, since 0 must be reachable from the right. Thus, by claim 4.2.1.2, for any finite, strongly connected S, $\beta_S \ge \min(\beta_{\{0,1,2\}}, \beta_{\{0,2\}})$. Now $\{0, 2\}$ and $\{0, 1, 2\}$ are already strongly connected, so $\kappa_0 = \min(\beta_{\{0,1,2\}}, \beta_{\{0,2\}})$. The minimum may be achieved on either set, depending on the α_i.

We now examine a case where $\kappa_0 \le 1$, but there are no loops that the walk is expected to iterate infinitely many times before doing anything else. Suppose $\alpha_{-2} = \alpha_2 = \frac{1}{9}$, and $\alpha_1 = \frac{1}{2}$. Then $\beta_{\{0,2\}} = \frac{11}{9} > 1$, and $\beta_{\{0,1,2\}} = \frac{17}{18} < 1$. Thus, $\kappa_0 = \frac{17}{18}$, and a walk started from 0 is expected to spend an infinite amount of time in $\{0, 1, 2\}$ before exiting. However, because $\beta_{\{0,2\}} = \frac{11}{9} > 1$, the expected exit time from $\{0, 2\}$ is finite, so the walk is not expected to iterate the loop $0 \to 2 \to 0$ infinitely many times before deviating from it. Moreover, the walk is not expected to iterate the loop $0 \to 1 \to 2 \to 0$ infinitely many times before deviating it, but to see this, we must use the original formulation of Tournier's lemma from [29]. The formulation there is in terms of sets of edges rather than vertices. The edges that are not in the loop $0 \to 1 \to 2$ but have tails in the vertex set touched by this loop have weights that add up to $\frac{19}{18} > 1$. Hence [29, Theorem 1] implies that the expected time to deviate from this set of edges (and thus from the loop $0 \to 1 \to 2 \to 0$) is finite. Nevertheless, the weight exiting $\{0, 1, 2\}$ is $\frac{17}{18} < \frac{1}{2}$, so the walk is expected to stick to the vertex set $\{0, 1, 2\}$, and thus to the pair of loops $0 \to 2 \to 0$ and $0 \to 1 \to 2 \to 0$, for an infinite amount of time before doing anything else. See Figure B.1.

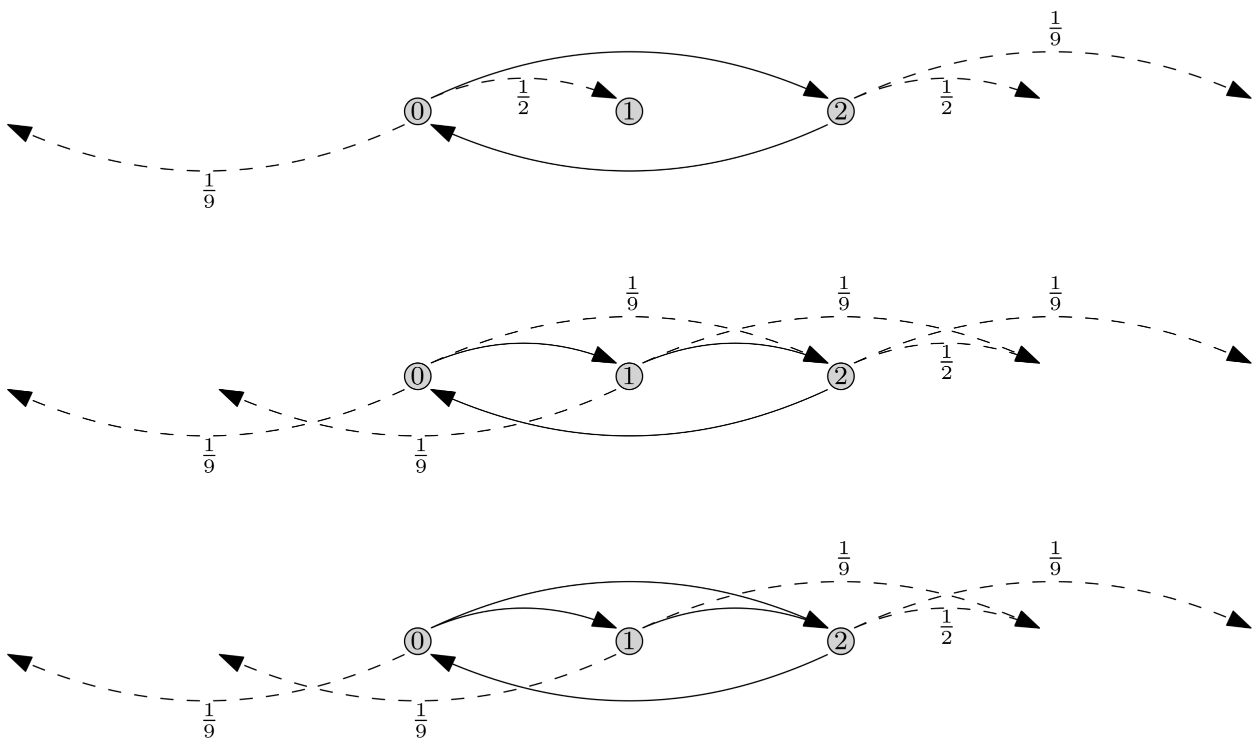

Figure B.1. The top shows the weights exiting the loop $0 \to 2 \to 0$. The middle shows the weights exiting the loop $0 \to 1 \to 2 \to 0$. The bottom shows the weights exiting the union of these two loops, or the set $\{0, 1, 2\}$.

Note that in this example, $\beta_{\{0,1,2\}} < \beta_{\{0,2\}}$. This shows that in Claim 4.2.1.2, the assumption that x is to the right or left of x is really needed.

Our next example presents a similar phenomenon: the walk is not expected to get stuck in any one loop for an infinite amount of time, but the walk is expected to spend an infinite amount of time in a set of vertices. In the previous example, the vertex set S minimizing β_S can have all of its vertices hit by one loop, the loop $0 \to 1 \to 2 \to 0$, but this loop alone does not have a trapping effect as strong as the whole set S. In the next example, there is no single loop that can hit all the vertices in the minimizing S, so our formulation of Tournier's lemma in terms of vertices only is enough to see that there is no single loop the walk is expected to traverse infinitely many times before straying from it. The next example also presents a calculation of κ_0 for a situation where it is less straightforward than the others we've examined.

Example B.0.8. $L = 16$, $R = 5$, $\alpha_{-16}, \alpha_2, \alpha_5 > 0$, all other $\alpha_i = 0$.

In this case, there are four possible values for κ_0, three of which can be attained by sets of vertices representing single loops, but one of which cannot. We will show that κ_0 is attained by one of the following four sets:

- $S_1 = \{0, 2, 4, 6, 8, 10, 12, 14, 16\}$. This set represents a loop that steps up by 2s from 0 to 16 and then jumps back to 0. $\beta_{S_1} = 8\alpha_{-16} + \alpha_2 + 9\alpha_5$.

- $S_2 = \{0, 5, 10, 15, 16, 20, 25, 30, 32\}$. The set S_2 represents a loop that steps up by 5s from 0 to 30, then steps to 32 and jumps back to 16 and then to 0. This is one of 28 loops that all step up by 5 six times, up by 2 once, and down by 16 twice, having vertex set S with leftmost point 0. All such loops have the same associated $\beta_S = \beta_{S_2} = 7\alpha_{-16} + 8\alpha_2 + 3\alpha_5$.

- $S_3 = \{0, 5, 10, 12, 14, 16\}$. The set S_3 represents a loop that steps up by 5s from 0 to 10, then by 2s from 0 to 16, then jumps back to 0. This is one of 10 loops that all have vertex set $S \subset [0, 16]$, all of which have $\beta_S = \beta_{S_3} = 5\alpha_{-16} + 3\alpha_2 + 4\alpha_5$.

- $S_4 = \{0, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16\}$. This set does *not* represent one single loop; in fact, it represents all 10 loops that stay within $[0, 16]$. $\beta_{S_4} = 12\alpha_{-16} + 2\alpha_2 + 5\alpha_5$.

One can check that any of β_{S_1}, β_{S_2}, β_{S_3}, and β_{S_4} can be the smallest, depending on the α_i. We will show

$$\kappa_0 = \min(\beta_{S_1}, \beta_{S_2}, \beta_{S_3}, \beta_{S_4}).$$

To confirm that these are the possible values for κ_0, let S be a finite, strongly connected set with leftmost point 0, and we will show that β_S is at least as large as one of these values. First, we note

$$\beta_S = x_{-16}\alpha_{-16} + x_2\alpha_2 + x_5\alpha_5,$$

where $x_i = x_i(S) := \#\{z \in S : z + i \notin S\} \geq 1$. Since 0 must be reachable in one step from a vertex to its right, $16 \in S$.

Claim B.0.8.1. $x_{-16} \geq 5$.

S must contain 0 and 16, so there must be a path σ from 0 to $[16, \infty)$ that does not leave S. Since the only step to the left is down 16, all steps in this path must be to the right (and must therefore be of length 2 or 5). If this path includes two or more steps of length 2, then $S \cap [0, 15]$ must have at least 5 elements. But for each $z \in S \cap [0, 15]$, $z - 16 \notin S$, so each element of $S \cap [0, 15]$ contributes 1 to x_{-16}. Hence $x_{-16} \geq 5$. On the other hand, if the path σ includes no steps or one step of length 2, then the path σ includes four vertices in $S \cap [0, 15]$, does not include 1 or 4, and lands on either 17 or 20. Since 1 and 4 are not in σ, either 17 or 20 contributes 1 to x_{-16}, in which case we then have $x_{-16} \geq 5$, or else 1 or 4 is in S, in addition to the four vertices from σ that are in $[0, 15]$, so that $|S \cap [0, 15]| \geq 5$, and so $x_{-16} \geq 5$. This proves our claim.

Claim B.0.8.2. If $x_2 = 1$, then $x_{-16} \geq 8$ and $x_5 \geq 9$.

To see this, note that if $x_2 = 1$, then S includes only even vertices; otherwise, the rightmost odd vertex and the rightmost even vertex would each contribute 1 to x_2, giving $x_2 \geq 2$. Moreover, since S contains 0 and 16, it must contain every even vertex between, in order to prevent any vertex other than the rightmost from contributing to x_2. The 8 even vertices $z = 0$ through $z = 14$ each have $z - 16 \notin S$, so $x_{-16} \geq 8$, and the 9 even vertices $z = 0$ through $z = 16$ each have $z + 5 \notin S$, so $x_5 \geq 9$.

Claim B.0.8.3. $x_5 \geq 3$.

To see this, note that the rightmost vertex from each equivalence class $\pmod 5$ will contribute 1 to x_5. We already have $0, 16 \in S$, so the equivalence classes 0 and 1 are represented. But the equivalence class 1 is only reachable from equivalence class 2 (via a downward step of length 16) and from equivalence class 4 (via an upward step of length 2). Hence S must contain an element from one of the equivalence classes 2 or 4 $\pmod 5$, and therefore at least three equivalence classes are represented, so $x_5 \geq 3$.

Claim B.0.8.4. If $x_5 = 3$, then $x_{-16} \geq 7$, and $x_2 \geq 8$.

We first note that since each equivalence class contributes only 1 to x_5, all elements in each equivalence class must form an unbroken arithmetic progression from the lowest to the highest. That is, letting z_i^{least} and z_i^{greatest} be, respectively, the least and greatest z such that $z \equiv i \pmod 5$ and $z \in S$, we have $\{z_i^{\text{least}}, z_i^{\text{least}} + 5, z_i^{\text{least}} + 10, \ldots, z_i^{\text{greatest}}\} \subset S$. We now examine two separate cases.

Case 1: S contains elements from equivalence classes 0,1, and 2 $\pmod 5$.

Since S contains no elements from equivalence class 4, equivalence z_1^{least} can only be reached from equivalence class 2, which occurs via a leftward step of length 16. Thus $z_1^{\text{least}} + 16 \in S$. On the other hand, since S contains no elements from equivalence class 3, equivalence class 2 can only be reached from equivalence class 0, via a rightward step of length 2. Thus $z_2^{\text{least}} - 2 \in S$.

Therefore, the path

$$0 \to 5 \to 10 \to \cdots \to (z_2^{\text{least}} - 2) \to z_2^{\text{least}} \to \cdots \to (z_1^{\text{least}} + 16) \to z_1^{\text{least}} \to \cdots \to 16 \to 0$$

is in S. All steps marked out by ellipses are upward steps of length 5. It follows that this is a path of length 9, since any other number of steps would result in ending at a point other than 0.

All the vertices in equivalence class 0 contribute 1 to x_{-16}, since stepping down by 16 would reach a vertex in equivalence class 4, which cannot be in S. Vertices from the path that are in equivalence class 2, other than $z_1^{\text{least}} + 16$, are less than $z_1^{\text{least}} + 16$, and so stepping down by 16 reaches a vertex that is in equivalence class 1 but not in S. And all vertices from the path that are in equivalence class 1, other than 16, are less than 16, so stepping down by 16 reaches a vertex not in S. Thus all but two of the vertices from the path shown will contribute 1 to x_{-16}, and therefore $x_{-16} \geq 7$.

Now all the vertices in equivalence classes 1 or 2 contribute 1 to x_2, since stepping to the right by 2 reaches a vertex in equivalence class 3 or 4. And vertices in equivalence class 0 that are less than $z_2^{\text{least}} - 2$ also contribute to x_2, since stepping to the right by 2 reaches a vertex in equivalence class 2 but less than z_2^{least}. Thus, all but one of the vertices shown in this path contribute to x_2, so $x_2 \geq 8$.

Case 2: S contains elements from equivalence classes 0,1, and 4 $\pmod 5$.

By a similar argument to that given in Case 1, the path

$$0 \to 5 \to 10 \to \cdots \to (z_4^{\text{least}} + 16) \to z_4^{\text{least}} \to \cdots \to (z_1^{\text{least}} - 2) \to z_1^{\text{least}} \to \cdots \to 16 \to 0$$

is in S. All steps marked out by ellipses are upward steps of length 5. It follows that this is a path of length 9, since any other number of steps would result in ending at a point other than 0. Now, for $z = 0, 5, 10, z_4^{\text{least}} + 11$, we have $z - 16 \equiv 4 \pmod 5$, but $z - 16 < z_4^{\text{least}}$, so $z - 16 \notin S$ and z contributes 1 to x_{-16}. Moreover, z_4^{least} and every subsequent vertex are all less than 16 (except, of course, for 16 itself), so they all contribute 1 to x_{-16}. Thus, $x_{-16} \geq 7$.

Moreover, all vertices in equivalence class 0 or 1 contribute 1 to x_2, since S has no vertices in equivalence class 2 or 4. And all but one of the vertices z in equivalence class 4 are strictly less than $z_1^{\text{least}} - 2$, so that $z + 2 \notin S$. Hence all but one of the vertices in the loop contribute to x_2, so $x_2 \geq 8$.

Claim B.0.8.5. If $x_2 = 2$, then $x_5 \geq 5$.

If $x_2 = 2$, then S contains even and odd elements (because the only even upward jumps are of length 2, a strongly connected S with only even elements would have $x_2 = 1$). It therefore must contain every even number, from its least even number to its greatest even number. In particular, it must contain $S_1 = [0, 16]$. This is enough to include at least one representative from every equivalence class $\pmod 5$. The greatest element of S in each of these equivalence classes contributes 1 to x_5, so $x_5 \geq 5$.

Claim B.0.8.6. If $x_2 = 2$, then $x_5 + x_{-16} \geq 17$ and $x_{-16} \geq 9$.

We have already established that if $x_2 = 2$, then S contains $S_1 = [0, 16]$ and at least one odd number. Now S_1 has $x_{-16} = 8$ and $x_5 = 9$. The odd number will also contribute 1 to x_{-16}, giving the bound $x_{-16} \geq 9$. The set S_1 includes 8, which is in equivalence class $3 \pmod 5$, and two elements of each of the equivalence classes 0,1,2, and 4. Each of the equivalence classes must contribute at least 1 to x_5, and for any of the classes 0,1,2, or 4 to avoid contributing 2, the odd number in between the two even numbers from that equivalence class must be contained in S. This saves 1 from x_5 but adds 1 to x_{-16}, thus keeping $x_5 + x_{-16} \geq 17$.

Claim B.0.8.7. $\beta_S \geq \min(\beta_{S_1}, \beta_{S_2}, \beta_{S_3}, \beta_{S_4})$.

We know β_S must have $x_5 \geq 3$ by Claim B.0.8.3. By Claim B.0.8.4, if $x_5 = 3$, then $\beta_S \geq 7\alpha_{-16} + 8\alpha_2 + 3\alpha_5 = \beta_{S_2}$. Now suppose $x_5 \geq 4$. If $x_2 = 1$, then $\beta_S \geq 8\alpha_{-16} + \alpha_2 + 9\alpha_5 = \beta_{S_1}$ by Claim B.0.8.2. Now consider the case $x_2 = 2$. Then $x_5 + x_{-16} \geq 17$ by Claim B.0.8.6. If $\alpha_5 > \alpha_{-16}$, then since $x_5 \geq 5$ by Claim B.0.8.5, we have $\beta_S \geq 12\alpha_{-16} + 2\alpha_2 + 5\alpha_5 = \beta_{S_4}$. On the other hand, if $\alpha_{-16} > \alpha_5$, then by Claim B.0.8.6, $\beta_S \geq 9\alpha_{-16} + 2\alpha_2 + 8\alpha_5 > 8\alpha_{-16} + \alpha_2 + 9\alpha_5 = \beta_{S_1}$. Now, if $x_2 \geq 3$, then by the assumption that $x_5 \geq 4$ and by Claim B.0.8.1, we have $\beta_S \geq 5\alpha_{-16} + 3\alpha_2 + 4\alpha_5 = \beta_{S_3}$. This proves our final claim.

Now suppose the weights are $\alpha_{-16} = \frac{1}{67}$, $\alpha_2 = \frac{15}{67}$, and $\alpha_5 = \frac{5}{67}$. We can check that $\kappa_0 = 12\alpha_{-16} + 2\alpha_2 + 5\alpha_5 = 1$, achieved on the set $S_4 = \{0, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16\}$, and that this is strictly less than β_{S_1}, β_{S_2}, and β_{S_3}. By the proof of Claim B.0.8.6, any set S with $x_{-16} = 12, x_2 = 2, x_5 = 5$ must contain a translation of $\{0, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16\}$, and a so there is no possibility that a set S which we did not consider, and which represents a single loop, *also* achieves $\beta_S = 1$. This means that the walk is expected to spend an infinite amount of time in the set $\{0, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16\}$ before ever leaving it, but there is no single loop that the walk is expected to take infinitely many times before deviating from it.

C. NOTATION

Here we collect notation that is used throughout the paper as a convenient reference.

General

- $\mathbb{N} = \{1, 2, 3, \ldots\}$. $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. $\mathbb{R}^{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$. $\mathbb{R}^{>0} = \{x \in \mathbb{R} : x > 0\}$.
- RWRE stands for random walk(s) in random environment(s). RWDE stands for random walk(s) in Dirichlet environment(s).
- An *environment* ω on a countable vertex set V, the set Ω_V of environments on V, and the measurable space $(\Omega_V, \mathcal{F}_V)$, are defined in Section 1.3.
- $\omega^x = (\omega(x, x+y))_{y \in \mathbb{Z}^d}$ is the environment ω viewed at site x only. For a set $S \subseteq \mathbb{Z}^d$, $\omega^S = (\omega^x)_{x \in S}$.
- An ω with a subscript (e.g., ω_1) or ω' is usually used to denote a specific environment when comparing multiple environments, and should not be confused with an ω with a superscript.
- Conditions (C1), (C2), and (C3) are defined in Section 1.3. They say the environments are i.i.d., almost surely irreducible, and almost surely have bounded jumps. Condition (C4) is defined in Section 2.3.1 for environments on $\mathbb{Z}$. It says the walk is almost surely transient to the right.
- A walk on the vertex set $\mathbb{Z}^d$ is a function from the set $\mathbb{N}_0$ of non-negative integers to the set $\mathbb{Z}^d$, denoted $\mathbf{X} = (X_n)_{n=0}^{\infty} \in (\mathbb{Z}^d)^{\mathbb{N}_0}$.
- $v := \lim_{n \to \infty} \frac{X_n}{n}$ is the $\mathbb{P}^0$ almost-sure limiting velocity that necessarily exists for all RWRE studied in this thesis.
- A walk is *transient to the right* if $\lim_{n \to \infty} X_n = \infty$, almost surely under the annealed measure.
- A walk is *ballistic* if $v > 0$.

- $\Delta_I := \{(x_i)_{i\in I} : \sum_{i\in I} x_i = 1\}$ is the simplex of a finite set I.
- θ^x is the left shift operator on environments ω, defined by $\theta^x\omega(a,b) = \omega(x+a, x+b)$.
- We use interval notation to denote sets of consecutive integers in the state space $\mathbb{Z}$, rather than subsets of $\mathbb{R}$. For example, $[1,\infty)$ denotes the set of integers to the right of 0. However, we make one exception, using $[0,1]$ to denote the set of all real numbers from 0 to 1.
- S^{d-1} is the unit sphere in $\mathbb{R}^d$.
- S_r^{d-1} is the set of $\ell \in S^{d-1}$ that have all rational slopes.
- The *annealed probability of* γ for a path $\gamma = (x_0, x_1, \ldots, x_n)$, is $\mathbb{P}^{x_0}(X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n)$.
- A *possible path* is a path that has positive annealed probability.

Graphs

- A weighted directed graph $\mathcal{H} = (V, E, W)$ is a vertex set V with an edge set $E \subseteq V \times V$, and a weight function $w : E \to \mathbb{R}^{>0}$.
- If $e = (x,y) \in E$, we say that e is an edge from x to y, and we say the *head* of e is $\overline{e} = y$ and the *tail* of e is $\underline{e} = x$.
- DERRW stands for directed edge reinforced random walk. For its definition and connection to RWDE, see page 18.
- For a vertex $x \in V$, the *divergence* of x in $\mathcal{H}$ is $\operatorname{div}(x) = \sum_{\overline{e}=x} w(e) - \sum_{\underline{e}=x} w(e)$. If the divergence is zero for all x, we say the graph $\mathcal{H}$ has zero divergence.
- A set $S \subset V$ is *strongly connected* if for all $x, y \in S$, there is a path from x to y in $\mathcal{H}$ using only vertices in S.
- For a set $S \subseteq V$, β_S is the sum of the weights of all edges exiting S. See (1.5).

- $\mathcal{N} \subset \mathbb{Z}^d$ is a set of allowed jumps. It obeys the condition that $\sum_{n=0}^{\infty}(\mathcal{N} \cup \{0\}) = \mathbb{Z}^d$ to ensure that the irreducibility condition (C2) is met.

- $\mathcal{G}$ is our "main graph." It has vertex set $\mathbb{Z}^d$, edge set $\{(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d : y - x \in \mathcal{N}\}$, and weight function $(x, y) \mapsto \alpha_{y-x}$.

- $\mathcal{G}_M$ is defined when $d = 1$. It is the finite graph with vertices $[0, M]$ that looks like $\mathcal{G}$ in the middle but is modified to have zero divergence at the endpoints. It is defined in the proof of the transient, one-dimensional case of Theorem 2.2.1. It can be used to prove Lemma 4.2.3.

- $\mathcal{G}_+$ is defined when $d = 1$. It is the half-infinite graph with vertex set $[0, \infty)$. It looks like $\mathcal{G}$ except near 0, where it looks like $\mathcal{G}_M$. It can be thought of as a limit of $\mathcal{G}_M$ as $M \to \infty$. It is defined at the beginning of Section 4.2.2. Its crucial property is given by Lemma 4.2.3, which is used in the proofs of Proposition 4.2.4, Proposition 4.2.5, and Theorem 2.3.5.

Parameters

- L and R are positive integers. The parameter L represents the maximum length of a jump to the left that has positive probability, and R represents the maximum length of a jump to the right.

- $(\alpha_i)_{i=-L}^{R}$ are Dirichlet parameters for random transition probability vectors. It is assumed that α_{-L} and α_R are positive.

- $d^+ = \sum_{i=1}^{R} i\alpha_i$, and $d^- = \sum_{i=-L}^{-1} |i|\alpha_i$.

- $c^+ = \sum_{i=1}^{R} \alpha_i$, and $c^- = \sum_{i=-L}^{-1} \alpha_i$.

- $\kappa_1 = d^+ - d^- = \sum_{i=-L}^{R} i\alpha_i$.

- $\kappa_0 = \inf\{\beta_S : S \subset \mathbb{Z}$ finite, strongly connected$\}$ is the minimum weight exiting a finite, strongly connected subset of $\mathcal{G}$.

- m_0 is an integer large enough that every interval of length m_0 is strongly connected in $\mathcal{G}$, and also large enough that $m_0 \geq \max(L, R)$.

Probability measures

- For a given environment ω on a vertex set V and a site $x \in V$, P_ω^x is the *quenched* probability measure on the set $V^{\mathbb{N}_0}$ where $P_\omega^0(X_0 = x) = 1$, and for $y \in V$, $P_\omega^x(X_{n+1} = y | X_0, \ldots, X_n) = \omega(X_n, y)$.

- P is a general probability measure on Ω_V.

- For $x \in V$, $\mathbb{P}^x = P \times P_\omega^x$ is the measure on $\Omega_V \times V^{\mathbb{N}_0}$ generated by $\mathbb{P}^x(A \times B) = \int_A P_\omega^x(B) P(d\omega)$. We abuse notation by also using $\mathbb{P}^x$ to refer to the marginal measure $\mathbb{P}^x(\Omega_V \times \cdot)$ on $V^{\mathbb{N}_0}$, which we call the *annealed* measure.

- For a given weighted directed graph $\mathcal{H} = (V, E, W)$, $P_\mathcal{H}$ is the Dirichlet law on environments corresponding to $\mathcal{H}$; that is, the measure on Ω_V under which transition probabilities at the various vertices $x \in V$ are independent, and for each vertex $x \in V$, $(\omega(x, \overline{e}))_{\underline{e}=x}$ is distributed according to a Dirichlet distribution with parameters $(w(e))_{\underline{e}=x}$. (Or, if $V \subset \mathbb{Z}$, we let $P_\mathcal{H}$ be any measure on $\Omega_\mathbb{Z}$ whose marginals on Ω_V are as described.)

- $P_\mathcal{G}$ is the main measure on $\Omega_\mathbb{Z}$ that we study in this paper. That is, $P_\mathcal{G}$ is the Dirichlet measure on environments corresponding to $\mathcal{G}$, so that for each x, the transition probability vector $(\omega(x, y))_{y \in \mathcal{N}}$ is distributed as a Dirichlet random vector with parameters $(\alpha_y)_{y \in \mathcal{N}}$.

Functions of a walk

For all of the functions below, we often suppress the argument of the function as is traditional with random variables. Sometimes, however, we leave the argument in when it is necessary for clarity.

- $T_x(\mathbf{X}) = \inf\{n \geq 0 : X_n = x\}$ is the first time the walk $(\mathbf{X})$ hits x. We often suppress the $\mathbf{X}$.

- $\tilde{T}_x(\mathbf{X}) = \inf\{n > 0 : X_n = x\}$ is the first nonzero time $\mathbf{X}$ hits x.

- $N_x(\mathbf{X}) = \#\{n \in \mathbb{N}_0 : X_n = x\}$ is the total amount of time $\mathbf{X}$ spends at x.

- For a subset $S \subset \mathbb{Z}$, $T_S(\mathbf{X}) = \min_{x\in S} T_x(\mathbf{X})$, and $N_S(\mathbf{X}) = \sum_{x\in S} N_x(\mathbf{X})$.

- $N_x^S(\mathbf{X}) = \#\{0 \leq n \leq T_{S^c} : X_n = x\}$ is the amount of time $\mathbf{X}$ spends at x before leaving the subset S.

- $N_{x,y}(\mathbf{X}) = \#\Big\{n \geq 0 : X_n = x, \sup\{k < n : X_k = y\} > \sup\{k < n : X_k = x\}\Big\}$ is the number of times the walk hits x after more recently having hit y, or the number of "trips from y to x."

- $N'_{x,y}(\mathbf{X}) := \#\Big\{n \in \mathbb{N}_0 : X_n \leq x, \sup\{j < n : X_j \geq y\} > \sup\{j < n : X_j \leq x\}\Big\}$ is the number of trips leftward across $[x, y]$. It is defined when $d = 1$.

- For any stopping time defined as an the first $n \geq 0$ such that satisfying a certain condition, we use the same notation but with a tilde ($\sim$) over it to denote the corresponding positive stopping time: that is, the first $n > 0$ satisfying the same condition.

Cascade and bi-infinite walk

Here, assume $d = 1$.

- A *cascade* is a set of finite (continuous-time) walks, one started at each point in $\mathbb{Z}$. The walk starting at each point terminates when it reaches or passes the next multiple of R.

- $\mathbf{X}^a = (X_n^a)_{n=0}^{\infty}$ is the infinite walk obtained by concatenating the finite walk starting at a with the walk starting at the point where it terminates, the walk started at the point where that one terminates, and so on.

- $\overline{\mathbf{X}} = (\overline{X}_n)_{n\in\mathbb{Z}}$ is the bi-infinite walk obtained by this process.

- $\overline{N}_x$, $\overline{T}_x$, $\overline{N}_{x,y}$, and so on are defined analogously to N_x, T_x, $N_{x,y}$, and so on, but with $n \geq 0$ replaced with $n \in \mathbb{Z}$.

- For a given environment ω, P_ω is the measure on the space of discrete-time cascades induced by ω, where the law of each $\mathbf{X}^a$ under P_ω is the law of $\mathbf{X}$ under P_ω^a.

- $\mathbb{P} = P \times P_\omega$ is the annealed measure on the space of environments and cascades.

Accelerated walks

- A continuous-time walk on $\mathbb{Z}^d$ is a function from the set $\mathbb{R}^{\geq 0}$ of non-negative reals to $\mathbb{Z}^d$, denoted $\mathbf{X} = (X_t)_{t \geq 0}$.

- A discrete-time walk may be thought of as a continuous-time walk where the position changes at integer times.

- A *Bouchet acceleration function* on $\mathbb{Z}^d$, defined on page 41, is a measurable function $\mathcal{A}$ from the space $\Omega_{\mathbb{Z}^d}$ of environments on $\mathbb{Z}^d$ to the space of distributions of positive random variables, where $\mathcal{A}(\omega)$ only depends on $\omega^{[-M,M]}$ for some positive integer M.

- $P_{\omega,\mathcal{A}}^x$ is the law of a continuous-time Markov chain started at x. Whenever the process hits a point $a \in \mathbb{Z}$, it remains there for an amount of time distributed according to $\mathcal{A}(\theta^a \omega)$ and independent of all other information about the history of the process before jumping to a point chosen according to ω.

- $\mathbb{P}_{\mathcal{A}}^x$ or $\mathbb{P}_{\mathcal{G},\mathcal{A}}^x$ is the corresponding annealed law.

- $v(\mathcal{A}) = \lim_{n\to\infty} \frac{X_t}{t}$ is the $\mathbb{P}_{\mathcal{A}}^0$ almost-sure limiting velocity that necessarily exists for all directionally transient RWRE on $\mathbb{Z}$.

- A measure P on $\Omega_{\mathbb{Z}^d}$ has *essential slowing* if, for any Bouchet acceleration function $\mathcal{A}$, it is the case that $v(\mathcal{A}) = 0$.

VITA

Daniel Slonim grew up in Chesterfield, Indiana. He was homeschooled by his mother along with three siblings from preschool through high school. He then attended Hillsdale College in Hillsdale, Michigan, where he earned a bachelor of science in mathematics and philosophy. While there, he conducted research through the LAUREATES program under the guidance of Will Abram, ultimately writing a math thesis on the topic of one-sided symbolic dynamics. He also graduated from Hillsdale's Honors Program, writing a thesis on the relationship between ethics and natural law. In 2016, he returned to his home state of Indiana for his graduate studies, entering the PhD program in mathematics at Purdue University. He focused primarily on discrete probability, but also continued working in symbolic dynamics. Along with Abram and Jeffrey Lagarias, he expanded the results of his undergraduate thesis into two papers, one of which has been published and one of which is in preparation. Daniel has written two papers on random walks in random environments, which have been submitted for publication. Daniel began the student probability seminar at Purdue. He also helped to organize the Bridge to Research seminar for a year, and served as a graduate representative for another year. After graduation, he plans to join the mathematics department at the University of Virginia as a Whyburn Research Associate and Lecturer.