1 Wednesday, August 23

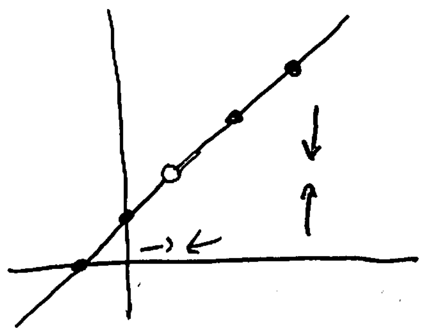
Example. Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

Examine the behavior of this function near $x = 1$.

Draw a graph:

x	$f(x)$
-1	0
0	1
1	DNE
2	3
3	4



The graph is a line.

$y = x + 1$.

Note: $f(x) = \dfrac{x^2-1}{x-1} = \dfrac{(x-1)(x+1)}{x-1} = x+1$

The value of $f(1)$ would have been 2 based on the behavior of $f(x)$ near $x=1$ if it had not been for the division by zero.

Example. Consider the piecewise-defined function

$$f(x) = \begin{cases} x+1 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

Examine the behavior of this function near $x = 1$.

Definition (Informal Definition of Limit). Suppose $f(x)$ is defined in an open interval about $x = c$, *except possibly at* $x = c$. If the values of $f(x)$ become arbitrarily close to L as the values of x approach c from both side,s then the **limit** of $f(x)$ as x approaches c is L, or

$$\lim_{x \to c} f(x) = L.$$

Note. In the definition of the limit, f need not be defined at $x = c$. In fact, if it is defined at $x = c$, then *ignore that.* The actual value of $f(c)$ has no bearing on the existence or value of $\lim_{x\to c} f(x)$; limits are an evaluation of the expected value of a function based on the values of nearby points. These limits are sometimes called **deleted limits** for this reason.

To find the limit of $f(x)$ as $x \to c$ numerically, simply evaluate the function at several points that grow closer to c. Then look to see if the values are growing closer to a specific number from either side.

Example. Estimate the following limits numerically.

(1) $\lim_{x\to 2} (3x - 5)$

1.9	1.99	1.999	2	2.001	2.01	2.1
0.7	0.97	0.997	–	1.003	1.03	1.3

$\lim_{x\to 2} (3x-5) = 1$

(2) $\lim_{x\to 4} (2x^2 - x + 1)$

3.9	3.99	3.999	4	4.001	4.01	4.1
27.52	28.8502	28.9850	–	29.0150	29.1502	30.52

$\lim_{x\to 4} (2x^2-x+1) = 29$

(3) $\lim_{x\to -1} (3x^3 - 1)$

(4) $\lim_{x \to -1} \frac{1}{(x+1)^2}$

-1.1	-1.01	-1.001	-1	-0.999	-0.99	-0.9
100	10,000	10^6	—	10^6	10,000	100

$$\lim_{x \to -1} \frac{1}{(x+1)^2} = \infty$$

(5) $\lim_{x \to 1} \frac{1}{(x+1)^2}$

(6) $\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 7x + 10}$

(7) $\lim_{x \to 0} \cancel{\frac{1 - \cos x}{x}}$ $\quad \frac{1 - \cos x}{x^2}$

Example. Consider the piecewise-defined function

$$f(x) = \begin{cases} x+2 & x<0 \\ x^2 & x \geq 0 \end{cases}$$

What is the limit as $x \to 0$?

One-Sided Limits

Definition (One-sided Limit). If $f(x) \to L$ as $x \to c$ from x values that are to the left of c ($x < c$), then

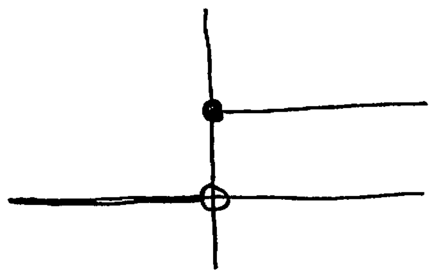
$$\lim_{x \to c^-} f(x) = L.$$

Similarly, if $f(x) \to M$ as $x \to c$ from x values that are to the right of c ($x > c$), then

$$\lim_{x \to c^+} f(x) = M.$$

Example. The step function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



x	-0.1	-0.01	0	0.01	0.1
$H(x)$	0	0	$-$	1	1

has a left hand limit of 0 and a right hand limit of 1 as $x \to 0$. However, the two-sided limit of this function does not exist since the two sides do not match.

$$\lim_{x \to 0^-} H(x) = 0$$

$$\lim_{x \to 0^+} H(x) = 1$$

$$\lim_{x \to 0} H(x) = \text{DNE}$$

Theorem (Two-sided Limit Existence).

$$\lim_{x \to c} f(x) = L \quad \text{if and only if} \quad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

Example. Consider the function

$$f(x) = \frac{2}{x^2 - 3x + 2}.$$

What are the one-sided limits of $f(x)$ as $x \to 2$? What is the two-sided limit of $f(x)$ as $x \to 2$?

Example. Consider the piecewise-defined function

$$f(x) = \begin{cases} x^2 & 0 \leq x < 2 \\ x - 1 & x \geq 2 \end{cases}$$

What are the one-sided limits of $f(x)$ as $x \to 2$? What is the two-sided limit of $f(x)$ as $x \to 2$?

Example. Consider the function

$$f(x) = -3x^2 + 1.$$

What are the one-sided limits of $f(x)$ as $x \to 0$? What is the two-sided limit of $f(x)$ as $x \to 0$?

Example. Consider the function

$$f(x) = \frac{6}{3 + e^{1/x}}.$$

What are the one-sided limits of $f(x)$ as $x \to 0$? What is the two-sided limit of $f(x)$ as $x \to 0$?

Example. Consider the piecewise-defined function

$$f(x) = \begin{cases} x^3 - x & x \leq 1 \\ (x-1)^2 & x > 1 \end{cases}$$

What are the one-sided limits of $f(x)$ as $x \to 1$? What is the two-sided limit of $f(x)$ as $x \to 1$?

Example. Consider the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

What are the one-sided limits of $f(x)$ as $x \to 0$? What is the two-sided limit of $f(x)$ as $x \to 0$?

0	0.00001	0.0001	0.001	0.01	0.1
–	0.0358	-0.3056	0.8269	-0.5064	-0.5440