Notes

$$x \longrightarrow \underbrace{\boxed{g} \longrightarrow u \longrightarrow \boxed{f}}_{f \circ g} \longrightarrow y$$

10 Friday, September 15

Chain Rule

Theorem 10.1 (Chain Rule). *Let $y = f(u)$ be differentiable with respect to u, and let $u = g(x)$ be differentiable with respect to x. Then $y = f(g(x))$ is differentiable with respect to x and*

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Note. The "u" in this definition is an "auxiliary" variable; when computing dy/dx, u should *not* appear in the final answer.

Example. Given y as a function of u and u as a function of x, find dy/dx.

(1) $y = u^3 + u - 1$
$u = 2x + 1$

$$\frac{dy}{du} = 3u^2 + 1 \qquad \frac{du}{dx} = 2$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (3u^2+1)(2) = 6u^2 + 2 = 6(2x+1)^2 + 2$$

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(2) $y = \sqrt{u}$
$u = x^2 + 2x - 6$

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} \qquad \frac{du}{dx} = 2x + 2$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2}u^{-1/2}(2x+2) = \frac{x+1}{u^{1/2}} = \frac{x+1}{\sqrt{x^2+2x-6}}$$

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(3) $y = \dfrac{1}{u+1}$
$u = x^3 - 2x + 5$

(4) $y = (x^3 + 2x^2 + x - 3)^5$

$y = u^5$
$u = x^3 + 2x^2 + x - 3$

$\frac{dy}{du} = 5u^4$

$\frac{du}{dx} = 3x^2 + 4x + 1$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$= 5u^4(3x^2 + 4x + 1)$$
$$= 5(x^3 + 2x^2 + x - 3)^4(3x^2 + 4x + 1)$$

Example. Suppose

$$f(2) = 8 \qquad f'(2) = \frac{1}{3} \qquad g(2) = 2 \qquad g'(2) = -3.$$

Find the derivative $h'(2)$.

(1) $h(x) = 2f(x)$

$h'(x) = 2f'(x)$ Constant Multiple Rule

$h'(2) = 2f'(2)$

$= \frac{2}{3}$

(2) $h(x) = f(x) + g(x)$

$h'(x) = f'(x) + g'(x)$ Addition Rule

$h'(2) = f'(2) + g'(2)$

$= \frac{1}{3} - 3$

$= -\frac{8}{3}$

(3) $h(x) = f(x)g(x)$

$h'(x) = f'(x)g(x) + f(x)g'(x)$ Product Rule

$h'(2) = f'(2)g(2) + f(2)g'(2)$

$= \left(\frac{1}{3}\right)(2) + (8)(-3)$

$= \frac{2}{3} - 24$

$= -\frac{70}{3}$

(4) $h(x) = f(x)/g(x)$

$h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$ Quotient Rule

$h'(2) = \dfrac{g(2)f'(2) - f(2)g'(2)}{g(2)^2}$

$= \dfrac{2\left(\frac{1}{3}\right) - 8(-3)}{4} = \dfrac{\frac{74}{3}}{4} = \dfrac{74}{12} = \dfrac{37}{6}$

(5) $h(x) = f(g(x))$

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(2) = f'(g(2)) \cdot g'(2)$

$= f'(2) \cdot g'(2)$

$= \frac{1}{3} \cdot (-3)$

$= -1$

(6) $h(x) = \sqrt{f(x)}$

$h = \sqrt{u}$

$u = f(x)$

$\frac{dh}{du} = \frac{1}{2} u^{-1/2}$

$\frac{du}{dx} = f'(x)$

$h'(x) = \frac{dh}{dx} = \frac{dh}{du} \frac{du}{dx}$

$= \frac{1}{2} u^{-1/2} \cdot f'(x)$

$h'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

$h'(2) = \frac{f'(2)}{2\sqrt{f(2)}} = \frac{\frac{1}{3}}{2\sqrt{8}} = \frac{1}{12\sqrt{2}}$

(7) $h(x) = 1/g(x)^2$

$h = \frac{1}{u^2}$

$u = g(x)$

$\frac{dh}{du} = -2u^{-3}$

$\frac{du}{dx} = g'(x)$

$h'(x) = \frac{dh}{dx} = \frac{dh}{du} \frac{du}{dx}$

$= -2u^{-3} \cdot g'(x)$

$= \frac{-2g'(x)}{g(x)^3}$

$h'(2) = \frac{-2g'(2)}{g(2)^3} = \frac{6}{8} = \frac{3}{4}$

(8) $h(x) = \sqrt{f(x)^2 + g(x)^2}$

$h'(x) = \frac{1}{2}\left[f(x)^2 + g(x)^2\right]^{-1/2} \cdot \left[2f(x)f'(x) + 2g(x)g'(x)\right]$

$h'(2) = \frac{1}{2}\left[f(2)^2 + g(2)^2\right]^{-1/2} \left[2f(2)f'(2) + 2g(2)g'(2)\right]$

$= \frac{1}{2}\left[64 + 4\right]^{-1/2} \left[\frac{16}{3} - 12\right]$

$= \frac{1}{2\sqrt{68}} \cdot \frac{-20}{3}$

$= \frac{-5}{3\sqrt{17}}$

Theorem 10.2 (General Power Rule). *If $f(x)$ is differentiable and n is a rational number, then*

$$\frac{d}{dx}[(f(x))^n] = nf(x)^{n-1}f'(x).$$

Example. Find $f'(x)$.

(1) $f(x) = (3x^2 - x + 1)^4$

$f'(x) = 4(3x^2 - x + 1)^3(6x - 1)$

(2) $f(x) = \sqrt[5]{x^4 - \frac{2}{x}}$

$f'(x) = \frac{1}{5}\left(x^4 - \frac{2}{x}\right)^{-4/5}\left(4x^3 + \frac{2}{x^2}\right)$

(3) $f(x) = \frac{1}{2x^2 - x + 5} = (2x^2 - x + 5)^{-1}$

$f'(x) = -(2x^2 - x + 5)^{-2}(4x - 1)$

(4) $f(x) = \frac{1}{(5x - 3)^6} = (5x - 3)^{-6}$

$f'(x) = -6(5x - 3)^{-7} \cdot (5)$

$= -30(5x - 3)^{-7}$

(5) $f(x) = \frac{1}{\sqrt{x^2-1}} \quad = (x^2-1)^{-1/2}$

$$f'(x) = -\frac{1}{2}(x^2-1)^{-3/2} \cdot (2x)$$

$$= \frac{-x}{(x^2-1)^{-3/2}}$$

(6) $f(x) = (10x-7)^6(x^2+1)^4$

$$f'(x) = \left[6(10x-7)^5(10)\right](x^2+1)^4 + (10x-7)^6\left[4(x^2+1)^3(2x)\right]$$

$$= 60(10x-7)^5(x^2+1)^4 + 8x(10x-7)^6(x^2+1)^3$$

$$= (10x-7)^5(x^2+1)^3\left[60(x^2+1) + 8x(10x-7)\right]$$

$$= (10x-7)^5(x^2+1)^3(140x^2-56x+60)$$

(7) $f(x) = \frac{(x-4)^3}{(2x+1)^7} = (x-4)^3(2x+1)^{-7}$

$$f'(x) = 3(x-4)^2(2x+1)^{-7} + (x-4)^3\left[-7(2x+1)^{-8}(2)\right]$$

$$= 3(x-4)^2(2x+1)^{-7} - 14(x-4)^3(2x+1)^{-8}$$

$$= (x-4)^2(2x+1)^{-8}\left[3(2x+1) - 14(x-4)\right]$$

$$= (x-4)^2(2x+1)^{-8}(-8x+59)$$

(8) $f(x) = \frac{\frac{1}{x} + x^2}{(2x+5)^3}$

$$f'(x) = \frac{(2x+5)^3\left(2x - \frac{1}{x^2}\right) - \left(\frac{1}{x} + x^2\right)\left[3(2x+5)^2(2)\right]}{(2x+5)^6}$$

$$= \frac{(2x+5)\left(2x - \frac{1}{x^2}\right) - 6\left(\frac{1}{x} + x^2\right)}{(2x+5)^4}$$

$$= \frac{-2x^2 + 10x - \frac{8}{x} - \frac{5}{x^2}}{(2x+5)^4}$$