Notes

11 Monday, September 18

Chain Rule

Theorem (Chain Rule). Let $y = f(u)$ be differentiable with respect to u, and let $u = g(x)$ be differentiable with respect to x. Then $y = f(g(x))$ is differentiable with respect to x and

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x) \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example (Using Chain Rule with Transcendental Functions). Find $f'(x)$.

(1) $f(x) = (\sin x - 1)^2$

$$f'(x) = 2(\sin x - 1)\cos x$$

(2) $f(x) = \sqrt[3]{1 + \cos(2x)}$

$$f'(x) = \frac{1}{3}(1+\cos(2x))^{-2/3}(-\sin(2x))(2)$$

$$= -\frac{2}{3}(1+\cos(2x))^{-2/3}\sin(2x)$$

(3) $f(x) = \sec(4x - 1)$

$$f'(x) = \sec(4x-1)\tan(4x-1)\cdot 4$$

$$= 4\sec(4x-1)\tan(4x-1).$$

$y = \sec u$

$u = 4x - 1$

$\frac{dy}{du} = \sec u \tan u$

$\frac{du}{dx} = 4$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= 4\sec u \tan u$$

$$= 4\sec(4x-1)\tan(4x-1)$$

(4) $f(x) = \tan\left[(2x+5)^{-2/3}\right]$

$$f'(x) = \sec^2\left((2x+5)^{-2/3}\right)\left[-\frac{2}{3}(2x+5)^{-5/3}\right](2)$$

$$= -\frac{4}{3}(2x+5)^{-5/3}\sec^2\left[(2x+5)^{-2/3}\right]$$

(5) $f(x) = \cos\left(e^{-x^2}\right)$

$$f'(x) = -\sin\left(e^{-x^2}\right)\cdot\left(e^{-x^2}\right)\cdot(-2x)$$

$$= 2xe^{-x^2}\sin\left(e^{-x^2}\right)$$

(6) $f(x) = e^{\sin x}(2x+1)$

$$f'(x) = e^{\sin x}(2) + (2x+1)\left(e^{\sin x}\cos x\right)$$

$$= e^{\sin x}\left(2 + 2x\cos x + \cos x\right)$$

(7) $f(x) = \tan(e^{4x})$

$$f'(x) = \sec^2(e^{4x}) \cdot e^{4x} \cdot 4$$
$$= 4e^{4x}\sec^2(e^{4x})$$

(8) $f(x) = e^{-x}\sin(3x)$

$$f'(x) = e^{-x}(3\cos(3x)) + (-e^{-x})\sin(3x)$$
$$= e^{-x}(3\cos(3x) - \sin(3x))$$

(9) $f(x) = (8 + \csc^2 x)^4$

$$f'(x) = 4(8+\csc^2 x)^3 (2\csc x)(-\csc x \cot x)$$
$$= -8(8+\csc^2 x)^3 \csc^2 x \cot x$$

Theorem 11.1 (Derivative of $\ln x$). *If u is a differentiable function of x, then*

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \qquad \frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx} = \frac{u'}{u}.$$

Example. Find $f'(x)$.

(1) $f(x) = \ln 3x$

$$f'(x) = \frac{1}{3x} \cdot 3 = \frac{1}{x}$$

(2) $f(x) = \ln kx$, k a constant

$$f'(x) = \frac{1}{Kx} \cdot K = \frac{1}{x}$$

$$\ln Kx = \underbrace{\ln K}_{\text{constant}} + \ln x$$

$$\frac{d}{dx} \ln kx = \frac{d}{dx}(\ln k + \ln x) = \frac{1}{x}$$

(3) $f(x) = (\ln x)^3$

$$f'(x) = \frac{3(\ln x)^2}{x}$$

(4) $f(x) = \frac{\ln x}{x}$

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x \cdot (1)}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

(5) $f(x) = \frac{\ln x}{1 + \ln x}$

$$f'(x) = \frac{(1+\ln x)\frac{1}{x} - (\ln x)\left(\frac{1}{x}\right)}{(1+\ln x)^2}$$

$$= \frac{\frac{1}{x}}{(1+\ln x)^2} = \frac{1}{x(1+\ln x)^2}$$

(6) $f(x) = \ln(\ln x)$

$$f'(x) = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$$

(7) $f(x) = \ln(\sec x + \tan x)$

$$f'(x) = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x)$$

$$= \frac{\sec x \, (\sec x + \tan x)}{\sec x + \tan x}$$

$$= \sec x$$

(8) $f(x) = \ln(3xe^{-x})$

$$f'(x) = \frac{1}{3xe^{-x}} \cdot (3e^{-x} - 3xe^{-x}) = \frac{3e^{-x}(1-x)}{3xe^{-x}} = \frac{1-x}{x}$$

$$= \frac{1}{x} - 1$$

$$f(x) = \ln 3 + \ln x + \ln(e^{-x}) = \ln 3 + \ln x - x$$

$$f'(x) = \frac{1}{x} - 1$$

(9) $f(x) = e^{\cos x + \ln x}$

$$f'(x) = e^{\cos x + \ln x} \cdot \left(-\sin x + \frac{1}{x}\right)$$

$$= e^{\cos x} e^{\ln x} \left(-\sin x + \frac{1}{x}\right)$$

$$= x e^{\cos x} \left(-\sin x + \frac{1}{x}\right)$$

$$= e^{\cos x} (1 - x \sin x)$$

(10) $f(x) = \ln\left(\frac{e^x}{1+e^x}\right) = \ln e^x - \ln(1+e^x)$

$$= x - \ln(1+e^x)$$

$$f'(x) = 1 - \frac{1}{1+e^x} \cdot e^x$$

$$= \frac{1+e^x-e^x}{1+e^x} = \frac{1}{1+e^x}$$

(11) $f(x) = \ln(\sec(\ln x))$

$$f'(x) = \frac{1}{\cancel{\sec(\ln x)}} \cdot \left[\cancel{\sec(\ln x)}\tan(\ln x)\right] \cdot \frac{1}{x}$$

$$= \frac{\tan(\ln x)}{x}$$

(12) $f(x) = \frac{1}{2}\ln\left(\cos^2(8x)\right)$

$$f'(x) = \frac{1}{\cancel{2}} \cdot \frac{1}{\cos^{\cancel{2}}(8x)} \cdot \left(\cancel{2}\cancel{\cos(8x)}\right)\left(-\sin(8x)\right)(8)$$

$$= \frac{-8\sin(8x)}{\cos(8x)}$$

$$= -8\tan(8x)$$