12 Wednesday, September 20

Higher Order Derivatives

Definition (Higher Order Derivatives). The derivative \( y' = \frac{dy}{dx} \) is the first (order) derivative of \( y \) with respect to \( x \). The derivative itself is a function of \( x \), and can be differentiated again to produce

\[
y'' = \frac{dy'}{dx} = \frac{d}{dx} \left[ \frac{dy}{dx} \right] = \frac{d^2y}{dx^2}.
\]

This is the second (order) derivative of \( y \). This process can be continued with the third derivative, fourth derivative, and so on. The arbitrary \( n \)th (order) derivative is denoted

\[
y^{(n)} = \frac{d^ny}{dx^n}.
\]

Example 12.1. Find the second and third derivatives.

(1) \( y = x^2 + x + 8 \)

\[
y' = 2x + 1
\]

\[
y'' = 2
\]

\[
y''' = 0
\]

(2) \( w = 3z^7 - 7z^3 + 21z^2 \)

\[
w' = 21z^6 - 21z^2 + 42z
\]

\[
w'' = 126z^5 - 42z + 42
\]

\[
w''' = 630z^4 - 42
\]
(3) \( y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \)

\[
\gamma' = -\frac{1}{2} x^{-\frac{3}{2}} \\
\gamma'' = \frac{3}{4} x^{-\frac{5}{2}} \\
\gamma''' = -\frac{15}{8} x^{-\frac{7}{2}} 
\]

(4) \( y = e^x \)

\[
\gamma' = e^x \\
\gamma'' = e^x \\
\gamma''' = e^x \\
\gamma'''' = e^x 
\]

(5) \( y = \sin x \)

\[
\gamma' = \cos x \\
\gamma'' = -\sin x \\
\gamma''' = -\cos x 
\]
\( y = \frac{\ln x}{e^x} \)

\[ y' = \frac{e^x \cdot \frac{1}{x} - (\ln x) e^x}{e^{2x}} = \frac{\frac{1}{x} - \ln x}{e^x} \]

\[ y'' = \frac{e^x \left(- \frac{1}{x^2} - \frac{1}{x} \right) - (\frac{1}{x} - \ln x) e^x}{e^{2x}} \]

\[ y'' = \frac{- \frac{1}{x^2} - \frac{3}{x} + \ln x}{e^x} \]

\[ y''' = \frac{e^x \left( \frac{3}{x^3} + \frac{3}{x^2} + \frac{1}{x} \right) - (\frac{1}{x^2} - \frac{3}{x} + \ln x) e^x}{e^{2x}} \]

\[ y''' = \frac{\frac{2}{x^3} + \frac{3}{x^2} + \frac{3}{x} - \ln x}{e^x} \]
Example 12.2. Find $y^{(101)}$.

(1) $y = x^3 - 2x + 1$
\[
y^{(f)} = 0 \quad \text{Note:} \quad \frac{d^m}{dx^m} x^n = 0 \quad \text{and} \quad \frac{d^m}{dx^m} x^m = 0 \quad \text{when} \quad m > n.
\]
\[
y^{(101)} = (101 \cdot 101 \cdot 100 \cdots 2)x - 101.
\]

(2) $y = x^{102} - x^{101} + x^{99} - 7$
\[
y^{(101)} = (101 \cdot 101 \cdot 100 \cdots 2)x - 101.
\]

(3) $y = e^x$
\[
y^{(1)} = 3e^x
\]
\[
y^{(101)} = 3e^x
\]

(4) $y = 2 \cos x$
\[
y^{(1)} = 2 \sin x
\]
\[
y^{(101)} = -2 \sin x
\]

(5) $y = \sin(3x)$
\[
y^{(1)} = 3 \cos (3x)
\]
\[
y^{(2)} = -9 \sin (3x) = -3^2 \sin (3x)
\]
\[
y^{(3)} = -27 \cos (3x) = -3^3 \cos (3x)
\]
\[
y^{(101)} = 81 \sin (3x) = 3^{10} \sin (3x)
\]
\[
y^{(101)} = 3^{10} \cos (3x)
\]

Derivatives cycle every $4$th derivative:
\[
y^{(n)} = \begin{cases} 
\frac{d}{dx} \cos x & n \equiv 0 \mod 4 \\
-\frac{d}{dx} \sin x & n \equiv 1 \mod 4 \\
-\frac{d}{dx} \cos x & n \equiv 2 \mod 4 \\
\frac{d}{dx} \sin x & n \equiv 3 \mod 4
\end{cases}
\]
\[
y^{(101)} = 3^{10} \sin (3x)
\]
Definition (Acceleration). Acceleration is the derivative of velocity with respect to time. If the position is given by \( s = f(t) \), then the acceleration is

\[
a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.
\]

Example 12.3.

(1) A rock is thrown vertically upward, reaching a height of \( s = 24t - 4.9t^2 \) meters after \( t \) seconds.

(a) Find the rock's velocity and acceleration at time \( t \).

\[
v(t) = \frac{ds}{dt} = 24 - 9.8t
\]

\[
a(t) = \frac{dv}{dt} = -9.8
\]

(b) How long would it take the rock to reach its highest point?

Rock is at highest point when \( v = 0 \):

\[
v(t) = 24 - 9.8t = 0
\]

\[
t = \frac{24}{9.8} = \frac{240}{98} = 2.4490 \text{ s}
\]

(c) How high would the rock go?

\[
s\left(\frac{240}{98}\right) = 24\left(\frac{240}{98}\right) - 4.9\left(\frac{240}{98}\right)^2
\]

\[
= 29.3878 \text{ m}
\]
(2) The position of an object is given by the function \( f(t) = 3e^{-t} \sin t \).

(a) Find the object's velocity and acceleration at time \( t \).

\[
v(t) = \frac{ds}{dt} = 3e^{-t} \cos t - 3e^{-t} \sin t
\]

\[= 3e^{-t} (\cos t - \sin t)\]

\[a(t) = \frac{dv}{dt} = 3e^{-t} (-\sin t - \cos t) - 3e^{-t} (\cos t - \sin t)
\]

\[= -6e^{-t} \cos t\]

(b) What is the object's velocity and acceleration at time \( t = \pi \)? \( t = \pi/4 \)?

\[v(\pi) = 3e^{-\pi} (\cos \pi - \sin \pi) = -3e^{-\pi}\]

\[v(\frac{\pi}{4}) = 3e^{-\pi/4} \left( \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = 0\]

\[a(\pi) = 6e^{-\pi}\]

\[a(\frac{\pi}{4}) = -6e^{-\pi/4} \cdot \frac{\sqrt{2}}{2} = -3\sqrt{2} e^{-\pi/4}\]

(c) When is the object at a standstill?

\[v(t) = 3e^{-t} (\cos t - \sin t) = 0\]

\[3e^{-t} = 0\]

\[\cos t - \sin t = 0\]

\[1 = \tan t\]

\[t = \frac{\pi}{4} + \pi n, \quad n \in \mathbb{Z}\]