12 Wednesday, September 20

Higher Order Derivatives

Definition (Higher Order Derivatives). The derivative $y' = dy/dx$ is the **first (order) derivative** of y with respect to x. The derivative itself is a function of x, and can be differentiated again to produce

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}.$$

This is the **second (order) derivative** of y. This process can be continued with the third derivative, fourth derivative, and so on. The arbitrary n**th (order) derivative** is denoted

$$y^{(n)} = \frac{d^ny}{dx^n}.$$

Example 12.1. Find the second and third derivatives.

(1) $y = x^2 + x + 8$

$y' = 2x + 1$

$y'' = 2$

$y''' = 0$

(2) $w = 3z^7 - 7z^3 + 21z^2$

$w' = 21z^6 - 21z^2 + 42z$

$w'' = 126z^5 - 42z + 42$

$w''' = 630z^4 - 42$

(3) $y = \frac{1}{\sqrt{x}} = x^{-1/2}$

$y' = -\frac{1}{2}x^{-3/2}$

$y'' = \frac{3}{4}x^{-5/2}$

$y''' = -\frac{15}{8}x^{-7/2}$

(4) $y = e^x$

$y' = e^x$

$y'' = e^x$

$y''' = c^x$

(5) $y = \sin x$

$y' = \cos x$

$y'' = -\sin x$

$y''' = -\cos x$

(6) $y = \frac{\ln x}{e^x}$

$$y' = \frac{e^x \cdot \frac{1}{x} - (\ln x)e^x}{e^{2x}} = \frac{\frac{1}{x} - \ln x}{e^x}$$

$$y'' = \frac{e^x\left(-\frac{1}{x^2} - \frac{1}{x}\right) - \left(\frac{1}{x} - \ln x\right)e^x}{e^{2x}}$$

$$= \frac{-\frac{1}{x^2} - \frac{2}{x} + \ln x}{e^x}$$

$$y''' = \frac{e^x\left(\frac{2}{x^3} + \frac{2}{x^2} + \frac{1}{x}\right) - \left(-\frac{1}{x^2} - \frac{2}{x} + \ln x\right)e^x}{e^{2x}}$$

$$= \frac{\frac{2}{x^3} + \frac{3}{x^2} + \frac{3}{x} - \ln x}{e^x}$$

Example 12.2. Find $y^{(101)}$.

(1) $y = x^3 - 2x + 1$

$y' = 3x^2 - 2$

$y'' = 6x$

$y''' = 6$

$y'''' = 0$

$y^{(5)} = 0$

$\vdots$

$y^{(101)} = 0$

Note:

$\frac{d^m}{dx^m} x^n = 0$ and

when $m > n$.

$\frac{d}{dx} x = 1$

$\frac{d^2}{dx} x^2 = 2$

$\frac{d^3}{dx^3} x^3 = 3 \cdot 2 \cdot 1 = 6$

$\vdots$

$\frac{d^n}{dx^n} x^n = n!$

factorial

(2) $y = x^{102} - x^{101} + x^{99} - 7$

$y^{(101)} = (102 \cdot 101 \cdot 100 \cdot \cdots \cdot 2)x - 101!$

(3) $y = 3e^x$

$y' = 3e^x$

$y'' = 3e^x$

$\vdots$

$y^{(101)} = 3e^x$

(4) $y = 2\cos x$

$y' = -2\sin x$

$y'' = -2\cos x$

$y''' = 2\sin x$

$y^{(4)} = 2\cos x$

Derivatives cycle every 4^{th} derivative:

$$y^{(n)} = \begin{cases} 2\cos x & n \equiv 0 \bmod 4 \\ -2\sin x & n \equiv 1 \bmod 4 \\ -2\cos x & n \equiv 2 \bmod 4 \\ 2\sin x & n \equiv 3 \bmod 4 \end{cases} \Rightarrow y^{(101)} = -2\sin x.$$

(5) $y = \sin(3x)$

$\rightarrow y' = 3\cos(3x)$

$y'' = -9\sin(3x) = -3^2\sin(3x)$

$y''' = -27\cos(3x) = -3^3\cos(3x)$

$y'''' = 81\sin(3x) = 3^4\sin(3x)$

$\vdots$

$y^{(101)} = 3^{101}\cos(3x)$

Definition (Acceleration). **Acceleration** is the derivative of velocity with respect to time. If the position is given by $s = f(t)$, then the acceleration is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Example 12.3.

(1) A rock is thrown vertically upward, reaching a height of $s = 24t - 4.9t^2$ meters after t seconds.

(a) Find the rock's velocity and acceleration at time t.

$$v(t) = \frac{ds}{dt} = 24 - 9.8t$$

$$a(t) = \frac{dv}{dt} = -9.8$$

(b) How long would it take the rock to reach its highest point?

Rock is at highest point when $v = 0$:

$$v(t) = 24 - 9.8t = 0$$

$$t = \frac{24}{9.8} = \frac{240}{98} = 2.4490 \text{ s}$$

(c) How high would the rock go?

$$s\left(\frac{240}{98}\right) = 24\left(\frac{240}{98}\right) - 4.9\left(\frac{240}{98}\right)^2$$

$$= 29.3878 \text{ m}$$

(2) The position of an object is given by the function $f(t) = 3e^{-t}\sin t$.

(a) Find the object's velocity and acceleration at time t.

$$v(t) = \frac{ds}{dt} = 3e^{-t}\cos t - 3e^{-t}\sin t$$
$$= 3e^{-t}(\cos t - \sin t)$$
$$a(t) = \frac{dv}{dt} = 3e^{-t}(-\cancel{\sin t} - \cos t) - 3e^{-t}(\cos t - \cancel{\sin t})$$
$$= -6e^{-t}\cos t$$

(b) What is the object's velocity and acceleration at time $t = \pi$? $t = \pi/4$?

$$v(\pi) = 3e^{-\pi}(-1)$$
$$= -3e^{-\pi}$$
$$a(\pi) = 6e^{-\pi}$$

$$v\left(\tfrac{\pi}{4}\right) = 3e^{-\pi/4}\left(\tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2}\right)$$
$$= 0$$
$$a\left(\tfrac{\pi}{4}\right) = -6e^{-\pi/4} \cdot \tfrac{\sqrt{2}}{2}$$
$$= -3\sqrt{2}\, e^{-\pi/4}$$

(c) When is the object at a standstill?

$$v(t) = 3e^{-t}(\cos t - \sin t) = 0$$

$$3e^{-t} = 0$$
$$\phi$$

$$\cos t - \sin t = 0$$
$$1 = \tan t$$
$$t = \tfrac{\pi}{4} + \pi n, \quad n \in \mathbb{Z}$$