Notes

13 Wednesday, September 27

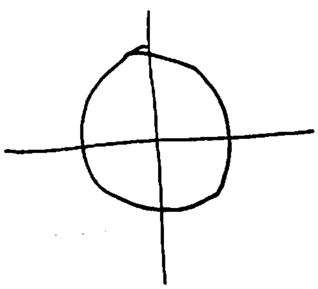
Implicit Differentiation

The derivative gives the slope of the tangent line on a graph given by a function. But not all graphs are represented by a function (i.e. graph does not pass vertical line test). However, these graphs still have tangent lines. The goal of this section is to find the slope of the tangent line in these graphs.

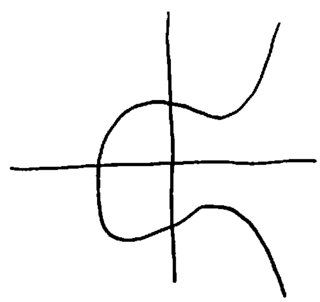
Definition. If $y = f(x)$, where $f(x)$ is just a function of x (no y's), then y is defined **explicitly**. Otherwise, y is defined **implicitly**.

Example. These equations define y implicitly.

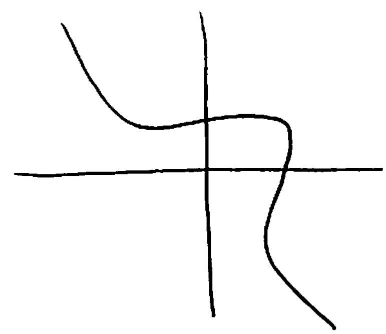
(1) $x^2 + y^2 = 1$



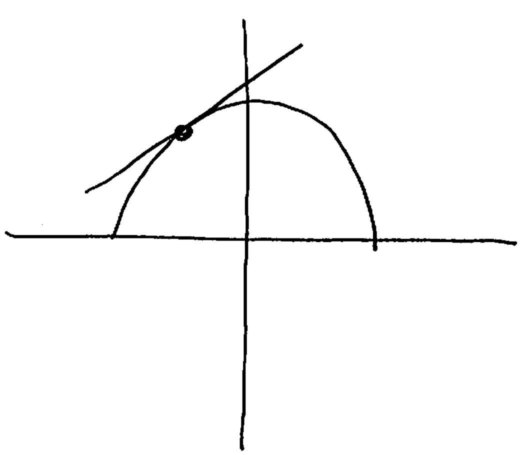
(2) $y^2 = x^3 - 2x + 2$

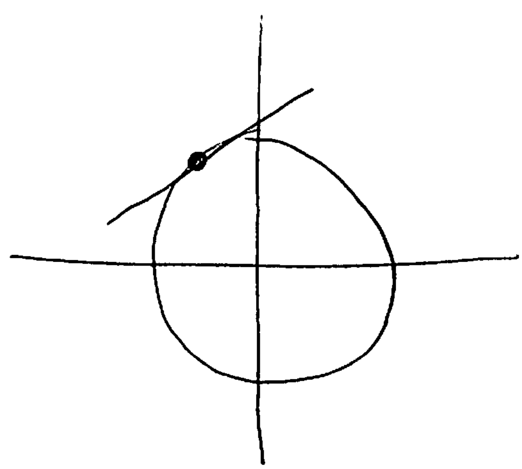


(3) $x^3 - xy + y^3 = 1$



Note. Implicit definitions do not necessarily define functions. However, functions can satisfy an implicit relation. For example, $y = f(x) = \sqrt{1-x^2}$ is a function that satisfies $x^2 + y^2 = 1$.





$y = f(x) = \sqrt{1-x^2}$

$x^2 + y^2 = 1$

$x^2 + (\sqrt{1-x^2})^2 = 1$

$1 = 1$ ✓

Example 13.1. Suppose $y = f(x)$ satisfies $x^2y + xy^2 = 6$. Find dy/dx. To do so, treat y like a function $f(x)$ and differentiate both sides, applying chain rule when appropriate.

Replace y with F(x).

$$x^2 F(x) + x[f(x)]^2 = 6$$

$$\frac{d}{dx}\left(x^2 f(x) + x[f(x)]^2\right) = \frac{d}{dx}(6)$$

$$x^2 f'(x) + 2x f(x) + [F(x)]^2 + 2x f(x) f'(x) = 0$$

$$x^2 y' + 2xy + y^2 + 2xyy' = 0$$

Solve for y'

$$x^2 y' + 2xyy' = -2xy - y^2$$

$$(x^2 + 2xy)y' = -2xy - y^2$$

$$y' = \frac{-2xy - y^2}{x^2 + 2xy}$$

Definition (Implicit Differentiation).

(1) Differentiate ~~the~~ equation treating y like $y(x)$, a function of x (this means that if y is in a term to be differentiated/ chain rule ~~or~~ product/quotient rule will be necessary).

(2) Solve for y'.

Implicit v. Explicit

$$x^2 + y^2 = 1$$

$$2x + 2yy' = 0$$

$$y' = -\frac{x}{y}$$

$$y = \sqrt{1 - x^2}$$

$$y' = \frac{1}{2}(1 - x^2)^{-1/2}(-2x)$$

$$= \frac{-x}{\sqrt{1 - x^2}}$$

$$= -\frac{x}{y}$$

Example 13.2. Find dy/dx.

(1) $2xy + y^2 = x + y$

$$\frac{d}{dx}(2xy + y^2) = \frac{d}{dx}(x+y)$$

$$2xy' + 2y + 2yy' = 1 + y'$$

$$2xy' + 2yy' - y' = 1 - 2y$$

$$(2x + 2y - 1)y' = 1 - 2y$$

$$y' = \frac{1-2y}{2x+2y-1}$$

(2) $x^2(x-y)^2 = x^2 - y^2$

$$\frac{d}{dx} x^2(x-y)^2 = \frac{d}{dx}(x^2 - y^2)$$

$$2x(x-y)^2 + 2x^2(x-y)(1-y') = 2x - 2yy'$$

$$2x(x-y)^2 + 2x^2(x-y) - 2x^2(x-y)y' = 2x - 2yy'$$

$$\frac{2x(x-y)^2 + 2x^2(x-y) - 2x = 2x^2(x-y)y' - 2yy'}{2}$$

$$x^3 - 2x^2y + xy^2 + x^3 - x^2y - x = (x^3 - x^2y - y)y'$$

$$y' = \frac{x^3 - x^2y - y}{2x^3 - 3x^2y + xy^2 - x}$$

(3) $(3xy+7)^2 = 6y$

$$\frac{d}{dx}(3xy+7)^2 = \frac{d}{dx}(6y)$$

$$2(3xy+7)(3y+3xy') = 6y'$$

$$\frac{18xy^2 + 42y + 18x^2yy' + 42xy' = 6y'}{6}$$

$$3xy^2 + 7y = y' - 3x^2yy' - 7xy'$$

$$3xy^2 + 7y = y'(1 - 3x^2y - 7x)$$

$$y' = \frac{3xy^2 + 7y}{1 - 3x^2y - 7x}$$

(4) $x^2 = \frac{x-y}{x+y}$

$$x^3 + x^2y = x - y$$

$$\frac{d}{dx}(x^3 + x^2y) = \frac{d}{dx}(x-y)$$

$$3x^2 + x^2y' + 2xy = 1 - y'$$

$$x^2y' + y' = 1 - 3x^2 - 2xy$$

$$y'(x^2+1) = 1 - 3x^2 - 2xy$$

$$y' = \frac{1 - 3x^2 - 2xy}{x^2+1}$$

(5) $x^3 - xy + y^3 = 1$

$$\frac{d}{dx}(x^3 - xy + y^3) = \frac{d}{dx}(1)$$

$$3x^2 - y - xy' + 3y^2 y' = 0$$

y' to the right side

y' to the left side

$$3x^2 - y = xy' - 3y^2 y'$$

$$3y^2 y' - xy' = y - 3x^2$$

$$y' = \frac{3x^2 - y}{x - 3y^2}$$

$$y' = \frac{y - 3x^2}{3y^2 - x}$$

these answers are the same when multiplied by $\frac{-1}{-1}$

(6) $(x^2 + 3y^2)^5 = 2xy$

$$\frac{d}{dx}(x^2 + 3y^2)^5 = \frac{d}{dx}(2xy)$$

$$5(x^2 + 3y^2)^4 (2x + 6yy') = 2y + 2xy'$$

$$\frac{10x(x^2 + 3y^2)^4 + 30yy'(x^2 + 3y^2)^4 = 2y + 2xy'}{2}$$

$$5x(x^2 + 3y^2)^4 - y = xy' - 15yy'(x^2 + 3y^2)^4$$

$$y' = \frac{5x(x^2 + 3y^2)^4 - y}{x - 15y(x^2 + 3y^2)^4}$$

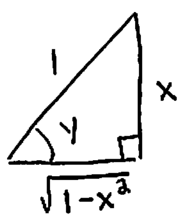
(7) $x = \sin y$

$$\frac{d}{dx} x = \frac{d}{dx} \sin y$$

$$1 = (\cos y) \cdot y'$$

$$y' = \frac{1}{\cos y}$$

Note: since $x = \sin y$, then $y = \sin^{-1} x = \arcsin x$, and



$$\cos y = \sqrt{1-x^2}$$

$$y' = \frac{1}{\sqrt{1-x^2}}$$, so $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$

(8) $x + \cos y = xy$

$$\frac{d}{dx}(x + \cos y) = \frac{d}{dx}(xy),$$

$$1 - y' \sin y = y + xy'$$

$$1 - y = y'(x + \sin y)$$

$$y' = \frac{1-y}{x + \sin y}$$

(9) $x \sin 2y = y \cos 2x$

$$\frac{d}{dx}(x \sin 2y) = \frac{d}{dx}(y \cos 2x)$$

$$\sin(2y) + x(\cos 2y)(2y') = y' \cos(2x) + y(-\sin(2x))(2)$$

$$\sin(2y) + 2y \sin(2x) = y' \cos(2x) - 2xy' \cos(2x)$$

$$y' = \frac{\sin(2y) + 2y \sin(2x)}{\cos(2x) - 2x \cos(2x)}$$

(10) $x = \ln y$ $\equiv$ $y = e^x$

$$\frac{d}{dx} x = \frac{d}{dx} \ln y$$

$$1 = \frac{1}{y} \cdot y'$$

$$y' = y$$

This is consistent with the derivative of $y = e^x$.

(11) $x + \ln(xy) = 0$

$$\frac{d}{dx}(x + \ln(xy)) = 0$$

$$1 + \frac{1}{xy}(y + xy') = 0$$

$$xy + y + xy' = 0$$

$$y' = \frac{-xy - y}{x}$$

(12) $\cos x + \cos y = xy$

$$\frac{d}{dx}(\cos x + \cos y) = \frac{d}{dx}(xy)$$

$$-\sin x - (\sin y)y' = y + xy'$$

$$-\sin x - y = y'\sin y + xy'$$

$$y' = \frac{-\sin x - y}{\sin y + x}$$

Example 13.3. Find the equation for the tangent line at the given point.

(1) $x^3 - y^3 = -19$ at $(2, 3)$

Slope of tangent is $\left.\frac{dy}{dx}\right|_{(x,y)=(2,3)}$:

$$\frac{d}{dx}(x^3 - y^3) = 0$$

$$3x^2 - 3y^2 y' = 0$$

$$y' = \frac{x^2}{y^2}$$

$$\left.\frac{dy}{dx}\right|_{(2,3)} = \frac{4}{9}$$

Point - Slope:

$$y - 3 = \frac{4}{9}(x - 2)$$

$$y = \frac{4}{9}x + \frac{19}{9}$$

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(2) $(2x + y)^3 = x$ at $(1, -1)$

Slope of tangent is $\left.\frac{dy}{dx}\right|_{(1,-1)}$

$$\frac{d}{dx}(2x + y)^3 = \frac{d}{dx}x$$

$$3(2x+y)^2(2 + y') = 1$$

(x,y) can be substituted at this stage:

$$3(2-1)^2(2 + y') = 1$$

$$2 + y' = \frac{1}{3}$$

$$y' = -\frac{5}{3}$$

Point - Slope:

$$y + 1 = -\frac{5}{3}(x - 1)$$

$$y = -\frac{5}{3}x + \frac{2}{3}$$

(3) $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at $(-1, 0)$

Slope of tangent is $\left.\frac{dy}{dx}\right|_{(-1,0)}$:

$$\frac{d}{dx}(6x^2 + 3xy + 2y^2 + 17y - 6) = 0$$

$$12x + 3y + 3xy' + 4yy' + 17y' = 0$$

At $(x,y) = (-1,0)$:

$$-12 + 0 - 3y' + 0 + 17y' = 0$$

$$y' = \frac{6}{7}$$

Point-Slope:

$$y = \frac{6}{7}(x+1)$$

$$= \frac{6}{7}x + \frac{6}{7}$$

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(4) $x = \tan y$ at $(1, \pi/4)$

Slope of tangent is $\left.\frac{dy}{dx}\right|_{(1,\frac{\pi}{4})}$:

$$\frac{d}{dx}x = \frac{d}{dx}\tan y$$

$$1 = y' \sec^2 y$$

$$y' = \frac{1}{\sec^2 y}$$

$$\left.\frac{dy}{dx}\right|_{(1,\frac{\pi}{4})} = \frac{1}{(\sqrt{2})^2} = \frac{1}{2}$$

Point-Slope:

$$y - \frac{\pi}{4} = \frac{1}{2}(x-1)$$

$$y = \frac{1}{2}x - \frac{1}{2} + \frac{\pi}{4}$$

(5) $2xy + \pi \sin y = 2\pi$ at $(1, \pi/2)$

Slope of tangent is $\left.\frac{dy}{dx}\right|_{(1,\frac{\pi}{2})}$

$$\frac{d}{dx}(2xy + \pi \sin y) = \frac{d}{dx}(2\pi)$$

$$2y + 2xy' + \pi y' \cos y = 0$$

At $(x,y) = (1, \frac{\pi}{2})$:

$$\pi + 2y' + \cancel{\pi y' \cos(\tfrac{\pi}{2})} = 0$$

$$y' = -\frac{\pi}{2}$$

Point - Slope:

$$y - \frac{\pi}{2} = -\frac{\pi}{2}(x-1)$$

$$y = -\frac{\pi}{2}x + \pi$$

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(6) $x^2 \cos^2 y - \sin y = 0$ at $(0, \pi)$

Slope of tangent is $\left.\frac{dy}{dx}\right|_{(0,\pi)}$:

$$\frac{d}{dx}(x^2 \cos^2 y - \sin y) = 0$$

$$2x \cos^2 y + x^2(2\cos y(-\sin y))y' - (\cos y)y' = 0$$

At $(x,y) = (0, \pi)$:

$$-y' \cos \pi = 0$$

$$y' = 0$$

Point - Slope:

$$y = \pi$$