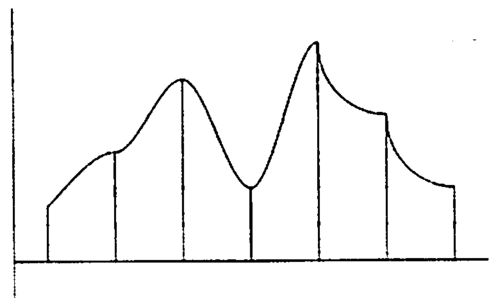
16 Wednesday, October 4

Relative Extrema and Critical Numbers

Definition (Relative Extrema). f has a **relative maximum** at $x = c$ if $f(c) \geq f(x)$ for all x in some interval (a, b) containing c. f has a **relative minimum** at $x = c$ if $f(c) \leq f(x)$ for all x in some interval (a, b) containing c. Relative extrema are also called **local extrema**.

Observation. Consider the following graph.



Note that points where there are extrema, the derivative of f is either 0 or does not exist.

Theorem 16.1 (First Derivative Test for Relative Extrema). *If f has a local maximum or local minimum at an interior point $x = c$, then either $f'(c) = 0$ or does not exist.*

Definition (Critical Numbers). If $f'(c) = 0$ or f is not differentiable at an interior point $x = c$, then c is a critical number.

Example. Find the critical numbers of each function in their domains.

(1) $f(x) = x^2 - 1$

$f'(x) = 2x = 0$

$\boxed{x = 0}$

(2) $f(x) = x - 5$

$f'(x) = 1 = 0$

$\boxed{\phi}$

(3) $f(x) = 8x^2 - x^4$

$f'(x) = 16x - 4x^3 = 4x(4 - x^2) = 0$

$\boxed{x = \pm 2, 0}$

(4) $f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3}x^{-2/3} = 0$ has no solution

But $f'(x)$ is DNE at $\boxed{x = 0}$ $(\div \text{ by } 0)$

$$2\cos(3x) + 2\sin(3x)\cos(3x)$$

(5) $f(x) = \cancel{2\cos x + \sin(2x)}$

$$f'(x) = -6\sin(3x) + 6\cos^2(3x) - 6\sin^2(3x)$$

$$0 = -6\sin(3x) + 6(1-\sin^2(3x)) - 6\sin^2(3x)$$

$$0 = -12\sin^2(3x) - 6\sin(3x) + 6$$

Let $y = \sin(3x)$.

$$-12y^2 - 6y + 6 = 0$$

$$2y^2 + y - 1 = 0$$

$$(2y-1)(y+1) = 0$$

$$y = \frac{1}{2}, -1$$

$\sin(3x) = -1$

$$3x = \frac{3\pi}{2} + 2\pi n, \quad n \in \mathbb{Z}$$

$$x = \frac{\pi}{2} + \frac{2\pi n}{3}, \quad n \in \mathbb{Z}$$

$\sin(3x) = \frac{1}{2}$

$$3x = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\} + 2\pi n, \quad n \in \mathbb{Z}$$

$$x = \left\{\frac{\pi}{18}, \frac{5\pi}{18}\right\} + \frac{2\pi}{3}n, \quad n \in \mathbb{Z}$$

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(6) $f(x) = \cot x + 2\csc x$

$$f'(x) = -\csc^2 x - 2\csc x \cot x$$

$$0 = -\csc x(\csc x + 2\cot x)$$

$-\csc x = 0$

$$-\frac{1}{\sin x} = 0$$

$$-1 = 0$$

$$\phi$$

$\csc x + 2\cot x = 0$

$$\frac{1}{\sin x} + \frac{2\cos x}{\sin x} = 0$$

$$1 + 2\cos x = 0$$

$$\cos x = -\frac{1}{2}$$

$$\boxed{x = \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} + 2\pi n, \quad n \in \mathbb{Z}}$$

There are points where $f'(x)$ does not exist:

$$\csc x = \frac{1}{\sin x} \qquad \cot x = \frac{\cos x}{\sin x}$$

These do not exist when

$$\sin x = 0 \rightarrow x = \pi n, \quad n \in \mathbb{Z}$$

But πn, $n \in \mathbb{Z}$ are not in the domain of $f(x)$, so they are not critical numbers.

(7) $f(x) = x\sqrt{4 - x^2}$

$$f'(x) = (4-x^2)^{1/2} + x\left[\frac{1}{2}(4-x^2)^{-1/2}(-2x)\right]$$

$$= \frac{4-x^2}{(4-x^2)^{1/2}} + \frac{-x^2}{(4-x^2)^{1/2}}$$

$$= \frac{4-2x^2}{(4-x^2)^{1/2}}$$

$$f'(x) = 0 = \frac{4-2x^2}{(4-x^2)^{1/2}}$$

$$0 = 4 - 2x^2$$

$$\boxed{x = \pm\sqrt{2}}$$

$$f'(x) = DNE = \frac{4-2x^2}{(4-x^2)^{1/2}}$$

$$0 = \sqrt{4-x^2}$$

$$\boxed{x = \pm 2}$$

(8) $f(x) = \dfrac{x}{x^2 - x + 1}$

$$f'(x) = \frac{(x^2-x+1)(1) - x(2x-1)}{(x^2-x+1)^2} = \frac{-x^2+1}{(x^2-x+1)^2}$$

$$f'(x) = 0 = \frac{-x^2+1}{(x^2-x+1)^2}$$

$$0 = -x^2 + 1$$

$$\boxed{x = \pm 1}$$

$$f'(x) = DNE = \frac{-x^2+1}{(x^2-x+1)^2}$$

$$0 = (x^2-x+1)^2$$

$$x = \frac{1 \pm \sqrt{1-4}}{2}$$

$$\boxed{\phi}$$

(9) $f(x) = \ln(x^2 + x + 1)$

$$f'(x) = \frac{2x+1}{x^2+x+1}$$

$$f'(x) = 0 = \frac{2x+1}{x^2+x+1}$$

$$0 = 2x+1$$

$$\boxed{x = -\frac{1}{2}}$$

$$f'(x) = \text{DNE} = \frac{2x+1}{x^2+x+1}$$

$$0 = x^2+x+1$$

$$x = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$\boxed{\phi}$$

(10) $f(x) = x^{3/4} - 2x^{1/4}$

$$f'(x) = \frac{3}{4}x^{-1/4} - \frac{1}{2}x^{-3/4} = \frac{1}{4}x^{-3/4}(3x^{1/2} - 2)$$

$$f'(x) = 0 = \frac{1}{4}x^{-3/4}(3x^{1/2} - 2)$$

$$0 = 3x^{1/2} - 2$$

$$\boxed{x = \frac{4}{9}}$$

$$f'(x) = \text{DNE} = \frac{1}{4}x^{-3/4}(3x^{1/2} - 2)$$

$$0 = 4x^{3/4}$$

$$\boxed{x = 0}$$

(11) $f(x) = x^{4/5}(x-4)^2$

$$f'(x) = \frac{4}{5}x^{-1/5}(x-4)^2 + x^{4/5}[2(x-4)]$$

$$= \frac{1}{5}x^{-1/5}(x-4)[4(x-4) + 10x]$$

$$= \frac{1}{5}x^{-1/5}(x-4)(14x-16)$$

$$f'(x) = 0 = \frac{1}{5}x^{-1/5}(x-4)(14x-16)$$

$$0 = (x-4)(14x-16)$$

$$\boxed{x = \frac{8}{7}, 4}$$

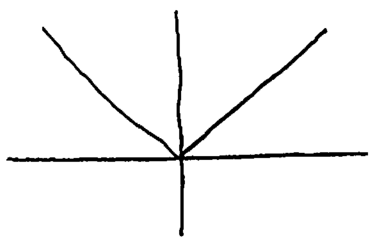
$$f'(x) = \text{DNE} = \frac{1}{5}x^{-1/5}(x-4)(14x-16)$$

$$0 = 5x^{1/5}$$

$$\boxed{x = 0}$$

(12) $f(x) = |x|$

the graph of $|x|$



shows that there is a corner at $x=0$.

This is a place where $f'(x) = DNE$, so $\boxed{x=0}$ is a critical number.

Analytically,

$$f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

and

$$f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

but the two-sided limit at $x=0$ does not exist, so $x=0$ is a critical number.

(13) $f(x) = |3x - 4|$

$$f(x) = |3x-4| = \begin{cases} 3x-4 & 3x-4 \geq 0 \equiv x \geq \frac{4}{3} \\ -(3x-4) & 3x-4 < 0 \equiv x < \frac{4}{3} \end{cases}$$

$$f'(x) = \begin{cases} 3 & x > \frac{4}{3} \\ -3 & x < \frac{4}{3} \end{cases}$$

$f'(x)$ does not exist at $\boxed{x = \frac{4}{3}}$

(14) $f(x) = xe^{-x^2/8}$

$$f'(x) = e^{-x^2/8} + x e^{-x^2/8}\left[-\frac{2x}{8}\right]$$

$$0 = e^{-x^2/8}\left[1 - \frac{1}{4}x^2\right]$$

$$e^{-x^2/8} = 0$$

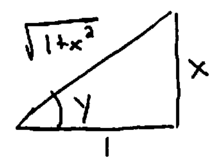
$$\emptyset$$

$$1 - \frac{1}{4}x^2 = 0$$

$$\boxed{x = \pm 2}$$

(15) $f(x) = x - 2\tan^{-1} x$

$$\tan y = x \rightarrow$$



$$y' \sec^2 y = 1$$

$$y' = \frac{1}{\sec^2 y}$$

$$y' = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}$$

$$f'(x) = 1 - \frac{2}{1+x^2}$$

$$= \frac{(1+x^2) - 2}{1+x^2}$$

$$= \frac{x^2-1}{x^2+1}$$

$$f'(x) = 0 = \frac{x^2-1}{x^2+1}$$

$$0 = x^2 - 1$$

$$\boxed{x = \pm 1}$$

$$f'(x) = DNE = \frac{x^2-1}{x^2+1}$$

$$0 = x^2 + 1$$

$$\boxed{\emptyset}$$