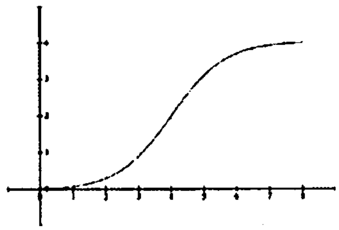
## 18 Friday, October 13

### Concavity

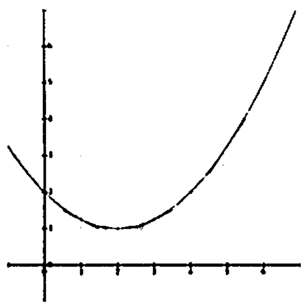
**Example.** Let $Q(t)$ be the number of units a worker had made after $t$ hours.



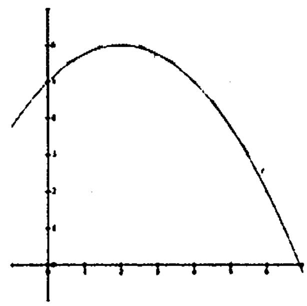
What is the time of his maximal efficiency, i.e. when is he producing units the fastest? His rate of production is the first derivative. To find the maximum of this, use the derivative of the derivative. If there is a maximum at a point, then by the FDT, the first derivative should change from increasing to decreasing.

**Definition** (Concavity). If $f$ is differentiable on $(a, b)$, then

(i) $f$ is concave up if $f'$ is increasing.



(ii) $f$ is concave down if $f'$ is decreasing.



**Example.** Find the concavity of $f(x) = x^4 - 6x^3 - 5$.

To find where $f'$ is inc/dec, find derivatives

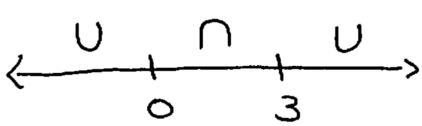
$$f'(x) = 4x^3 - 18x^2 = 2x^2(2x-9)$$

$$f''(x) = 12x^2 - 36x = 12x(x-3)$$

Find crit #'s of $f'$ ($f''=0$ or DNE)

$$f''(x) = 12x(x-3) = 0 \quad \rightarrow \quad x = 0, 3$$

Plot on number line




CU: $(-\infty, 0) \cup (3, \infty)$

CD: $(0, 3)$

$f''(-1) = +$ $\qquad$ $f''(4) = +$

$f''(1) = -$

**Theorem** (Second Derivative Test for Concavity). Suppose $f$ is twice differentiable on $(a, b)$.

(i) If $f''(x) > 0$ for all $x$ in $(a, b)$, then $f$ is concave up.

(ii) If $f''(x) < 0$ for all $x$ in $(a, b)$, then $f$ is concave down.

**Definition** (Inflection Points). A point on the graph of $f$ where $f$ is continuous and the concavity changes.

Finding Intervals of Concavity and Inflection Points:

(i) Find $f''$ and the critical numbers of $f'$. Mark them on a number line.

(ii) Pick test points from each interval and determine the sign of $f''$ at those points.

(iii) Apply the SDT to determine concavity. In addition, any critical number of $f'$ where the concavity changes and $f$ is continuous is an inflection point.

**Example.** Find inflection points of $f(x) = x^4 - 6x^3 - 5$.

Concavity

$\cup \quad \cap \quad \cup$

$0 \quad 3$

IP IP

Concavity changes at $x = 0, 3$

**Theorem** (Second Derivative Test for Relative Extrema). Suppose $f'(c) = 0$ and $f''(x)$ exists on $(a, b)$ containing $c$. Then

(i) If $f''(c) < 0$, then $f$ has a relative maximum at $x = c$.

(ii) If $f''(c) > 0$, then $f$ has a relative minimum at $x = c$.

(iii) If $f''(c) = 0$, then the test is inconclusive and FDT must be used.

**Note.** In the second derivative test for relative extrema, an inconclusive result *does not* imply no extrema occur at that point. See the above example.1

**Example 18.1.** Find the intervals of increasing/decreasing, concavity, inflection points, and relative extrema for each function.

(1) $f(x) = x^5 - 5x^4$

Derivatives:

$f'(x) = 5x^4 - 20x^3 = 5x^3(x-4)$

$f''(x) = 20x^3 - 60x^2 = 20x^2(x-3)$

Inc/Dec:

- Crit #'s of $f$ ($f' = 0$ or DNE)

  $f'(x) = 5x^3(x-4) = 0 \rightarrow x = 0, 4$

- Plot

  $\nearrow \quad \searrow \quad \nearrow$

  sign $f'$ $\quad + \quad 0 \quad - \quad 4 \quad +$

  $\boxed{\text{I: } (-\infty, 0) \cup (4, \infty) \quad \text{D: } (0, 4)}$

Extrema:

$x = 0$

$f''(0) = 0$ SDT inconclusive

$\nearrow \quad \searrow$

0

$\boxed{\text{FDT max}}$

$x = 4$

$f''(4) = +$

$\boxed{\text{SDT min}}$

Concavity:

- Crit #'s of $f'$ ($f'' = 0$ or DNE)

  $f''(x) = 20x^2(x-3) = 0 \rightarrow x = 0, 3$

- Plot

  $\cap \quad \cap \quad \cup$

  sign $f''$ $\quad - \quad 0 \quad - \quad 3 \quad +$

  $\boxed{\text{CU: } (3, \infty) \quad \text{CD: } (-\infty, 3)}$

Inflection:

$x = 0$ no concavity change

$\boxed{x = 3 \quad \text{concavity change}}$

(2) $f(x) = x^{1/5}$

Derivatives:

$f'(x) = \frac{1}{5} x^{-4/5}$

$f''(x) = -\frac{4}{25} x^{-9/5}$

Inc/Dec

- Crit #'s of $f$ ($f' = 0$ or DNE)

  $f'(x) = \frac{1}{5} x^{-4/5} \rightarrow x = 0$

- Plot

  $\nearrow \quad \nearrow$

  sign $f'$ $\quad + \quad 0 \quad +$

  $\boxed{\text{I: } (-\infty, \infty) \quad \text{D: NONE}}$

Extrema:

$x = 0$

$f''(0) = \text{DNE}$ SDT inconclusive

$\nearrow \quad \nearrow$

0

$\boxed{\text{FDT no extremum}}$

Concavity

- Crit #'s of $f'$ ($f'' = 0$ or DNE)

  $f''(x) = -\frac{4}{25} x^{-9/5} \rightarrow x = 0$

- Plot

  $\cup \quad \cap$

  sign $f''$ $\quad + \quad 0 \quad -$

  $\boxed{\text{CU: } (-\infty, 0) \quad \text{CD: } (0, \infty)}$

Inflection:

$\boxed{x = 0 \quad \text{concavity change}}$

(3) $f(x) = (x+1)(x-1)^2$

Derivatives:

$$f'(x) = (1)(x-1)^2 + (x+1)\left[2(x-1)\right] = (x-1)\left[(x-1)+2(x+1)\right] = (x-1)(3x+1)$$

$$f''(x) = (1)(3x+1) + (x-1)(3) = 6x - 2$$

Inc / Dec:

- Crit #'s of f ($f'=0$ or DNE)

  $f'(x) = (x-1)(3x+1) = 0 \rightarrow x = -\frac{1}{3}, 1$

- Plot

  $\nearrow \quad \searrow \quad \nearrow$

  sign $f'$ $\quad + \quad -\frac{1}{3} \quad - \quad 1 \quad +$

$$\boxed{\begin{array}{l} I: (-\infty, -\frac{1}{3}) \cup (1, \infty) \\ D: (-\frac{1}{3}, 1) \end{array}}$$

Extrema:

$x = -\frac{1}{3}$

$f''(-\frac{1}{3}) = -$ $\quad \boxed{\text{SDT max}}$

$x = 1$

$f''(1) = +$ $\quad \boxed{\text{SDT min}}$

Concavity:

- Crit #'s of f' ($f''=0$ or DNE)

  $f''(x) = 6x - 2 = 0 \rightarrow x = \frac{1}{3}$

- Plot

  $\cap \quad \cup$

  sign $f''$ $\quad - \quad \frac{1}{3} \quad +$

$$\boxed{\begin{array}{l} CU: (\frac{1}{3}, \infty) \\ CD: (-\infty, \frac{1}{3}) \end{array}}$$

Inflection:

$\boxed{x = \frac{1}{3} \quad \text{concavity changes}}$

(4) $f(x) = \dfrac{x^2-3}{x-2}$

Derivatives:

$$f'(x) = \frac{(x-2)(2x)-(x^2-3)}{(x-2)^2} = \frac{x^2-4x+3}{(x-2)^2} = \frac{(x-1)(x-3)}{(x-2)^2}$$

$$f''(x) = \frac{(x-2)^2(2x-4)-(x^2-4x+3)\left[2(x-2)\right]}{(x-2)^4} = \frac{2}{(x-2)^3}$$

Inc/Dec:

- Crit #'s of f ($f'=0$ or DNE)

$$f'(x) = \frac{(x-1)(x-3)}{(x-2)^2} \quad \rightarrow \quad x = 1, \cancel{2}, 3$$

not in domain.

- Plot

↗ ↘ ↘ ↗

sign f' + 1 − 2 − 3 +

$\boxed{\begin{array}{l} I: (-\infty,1)\cup(3,\infty) \\ D: (1,2)\cup(2,3) \end{array}}$

Extrema:

$x=1$:

$f''(1) = -$ $\boxed{\text{SDT max}}$

$x=3$:

$f''(3) = +$ $\boxed{\text{SDT min}}$

Concavity:

- Crit #'s of f' ($f''=0$ or DNE)

$$f''(x) = \frac{2}{(x-2)^3} \quad \rightarrow \quad x = \cancel{2}$$

not in domain

- Plot

∩ ∪

sign f'' − 2 +

$\boxed{\begin{array}{l} CU: (2,\infty) \\ CD: (-\infty,2) \end{array}}$

Inflection:

$\boxed{\text{none}}$

(5) $f(x) = x\sqrt{8-x^2}$ Domain: $[-2\sqrt{2}, 2\sqrt{2}]$

Derivatives:

$$f'(x) = (8-x^2)^{1/2} + x\left[\tfrac{1}{2}(8-x^2)^{-1/2}(-2x)\right] = \frac{8-2x^2}{(8-x^2)^{1/2}}$$

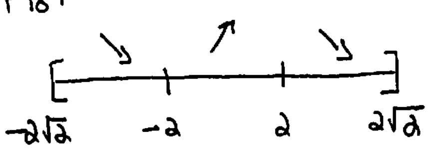
$$f''(x) = \frac{(8-x^2)^{1/2}(-4x) - (8-2x^2)\left[\tfrac{1}{2}(8-x^2)^{-1/2}(-2x)\right]}{(8-x^2)} = \frac{2x(x^2-12)}{(8-x^2)^{3/2}}$$

Inc/Dec:

- Crit #'s of f ($f'=0$ or DNE)

$$f'(x) = \frac{8-2x^2}{(8-x^2)^{1/2}} \quad \rightarrow \quad x = \pm 2\sqrt{2}, \pm 2$$

- Plot




sign $f'$ − + −

$\boxed{I: (-2,2) \quad D: [-2\sqrt{2},-2) \cup (2, 2\sqrt{2}]}$

Extrema:

$\boxed{x = -2\sqrt{2} \quad \text{max}}$ $\boxed{x = 2\sqrt{2} \quad \text{min}}$

$x = -2$ $x = 2$

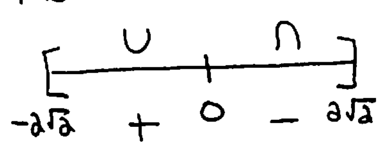
$f''(-2) = +$ $\boxed{\text{SDT min}}$ $f''(2) = -$ $\boxed{\text{SDT max}}$

Concavity:

- Crit #'s of $f'$ ($f''=0$ or DNE) not in domain

$$f''(x) = \frac{2x(x^2-12)}{(8-x^2)^{3/2}} \quad \rightarrow \quad x = \pm\cancel{2\sqrt{3}}, \pm 2\sqrt{2}, 0$$

- Plot




sign $f''$ + −

$\boxed{CU: [-2\sqrt{2}, 0) \quad CD: (0, 2\sqrt{2}]}$

Inflection:

$\boxed{x = 0 \quad \text{concavity changes}}$

(6) $f(x) = \frac{x+1}{(x-1)^2}$

Derivatives

$$f'(x) = \frac{(x-1)^2(1) - (x+1)[2(x-1)]}{(x-1)^4} = \frac{-x-3}{(x-1)^3}$$

$$f''(x) = \frac{(x-1)^3(-1) - (-x-3)[3(x-1)^2]}{(x-1)^6} = \frac{2x+10}{(x-1)^4}$$

Inc / Dec:

- Crit #'s of f ($f'=0$ or DNE)

$$f'(x) = \frac{-x-3}{(x-1)^3} \to x = -3, \cancel{1}$$

not in domain

- Plot

↘ -3 ↗ 1 ↘

sign $f'$: $-$ | $-3$ | $+$ | $1$ | $-$

$$\boxed{\begin{array}{l} I: (-3, 1) \\ D: (-\infty, -3) \cup (1, \infty) \end{array}}$$

Extrema:

$x = -3$

$f''(-3) = +$ $\boxed{\text{SDT min}}$

Concavity:

- Crit #'s of $f'$ ($f''=0$ or DNE)

$$f''(x) = \frac{2x+10}{(x-1)^4} \to x = -5, \cancel{1}$$

not in domain

- Plot

∪ -5 ∩ 1 ∪

sign $f''$: $+$ | $-5$ | $-$ | $1$ | $+$

$$\boxed{\begin{array}{l} CU: (-\infty, -5) \cup (1, \infty) \\ CD: (-5, 1) \end{array}}$$

Inflection:

$\boxed{x = -5 \quad \text{concavity changes}}$

(7) $f(x) = \frac{12}{x^2+12} \quad = \quad 12\left(x^2+12\right)^{-1}$

Derivatives

$$f'(x) = \frac{-24x}{(x^2+12)^2}$$

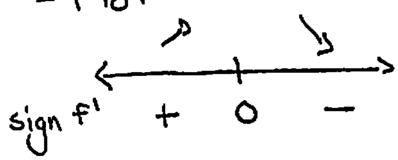
$$f''(x) = \frac{(x^2+12)^2(-24) - (-24x)\left[2(x^2+12)(2x)\right]}{(x^2+12)^4} = \frac{72(x^2-4)}{(x^2+12)^3}$$

Inc / Dec:

- Crit #'s of f ($f'=0$ or DNE)

$$f'(x) = \frac{-24x}{(x^2+12)^2} \quad \longrightarrow \quad x = 0$$

- Plot




$$\boxed{\begin{array}{l} I: (-\infty, 0) \\ D: (0, \infty) \end{array}}$$

Extrema:

$x = 0$

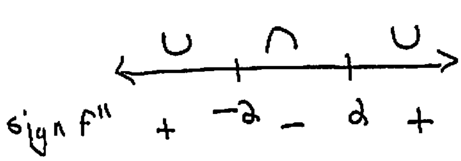
$f''(0) = -$ $\boxed{\text{SDT max}}$

Concavity:

- Crit #'s of f' ($f''=0$ or DNE)

$$f''(x) = \frac{72(x^2-4)}{(x^2+12)^3} \quad \longrightarrow \quad x = \pm 2$$

- Plot




$$\boxed{\begin{array}{l} CU: (-\infty, -2) \cup (2, \infty) \\ CD: (-2, 2) \end{array}}$$

Inflection:

$\boxed{x = \pm 2 \quad \text{concavity changes}}$

(8) $f(x) = x + \frac{9}{x}$

Derivatives:

$$f'(x) = 1 - \frac{9}{x^2}$$

$$f''(x) = \frac{18}{x^3}$$

Inc / Dec

- Crit #'s of f ($f'=0$ or DNE)

$f'(x) = 1 - \frac{9}{x^2} = 0 \;\rightarrow\; x^2 = 9 \;\rightarrow\; x = \pm 3$

$f'(x) = \text{DNE} \;\rightarrow\; x = \cancel{0}$ not in domain

- Plot

sign $f'$ $+$ $-3$ $-$ $0$ $-$ $3$ $+$

$$\boxed{\begin{array}{l} I: (-\infty, -3) \cup (3, \infty) \\ D: (-3, 0) \cup (0, 3) \end{array}}$$

Extrema:

$x = -3$

$f''(-3) = -$ $\boxed{\text{SDT max}}$

$x = 3$

$f''(3) = +$ $\boxed{\text{SDT min}}$

Concavity

- Crit #'s of f' ($f''=0$ or DNE)

$f''(x) = \frac{18}{x^3} \;\rightarrow\; x = \cancel{0}$ not in domain

- Plot

$\cap$ $\cup$

sign $f''$ $-$ $0$ $+$

$$\boxed{\begin{array}{l} CU: (0, \infty) \\ CD: (-\infty, 0) \end{array}}$$

Inflection:

$\boxed{\text{none}}$

(9) $f(x) = (2-x^2)^{3/2}$ Domain: $[-\sqrt{2}, \sqrt{2}]$

Derivatives:

$$f'(x) = \frac{3}{2}(2-x^2)^{1/2}(-2x) = -3x(2-x^2)^{1/2}$$

$$f''(x) = -3(2-x^2)^{1/2} + (-3x)\left[\frac{1}{2}(2-x^2)^{-1/2}(-2x)\right] = \frac{6x^2-6}{(2-x^2)^{1/2}}$$

Inc / Dec:

- Crit #'s of f ($f'=0$ or DNE)

$$f'(x) = -3x(2-x^2)^{1/2} \rightarrow x = \pm\sqrt{2}, 0$$

- Plot

$\nearrow$ $\searrow$

$[-\sqrt{2}$ $\quad 0 \quad$ $\sqrt{2}]$

sign $f'$ $\quad +$ $\quad -$

$\boxed{\begin{array}{l} I: [-\sqrt{2}, 0) \\ D: (0, \sqrt{2}] \end{array}}$

Extrema:

$\boxed{x = -\sqrt{2} \quad \text{min}}$ $\qquad$ $\boxed{x = \sqrt{2} \quad \text{min}}$

$x = 0$

$f''(0) = -$ $\boxed{\text{SDT max}}$

Concavity:

- Crit #'s of $f'$ ($f''=0$ or DNE)

$$f''(x) = \frac{6x^2-6}{(2-x^2)^{1/2}} \rightarrow x = \pm\sqrt{2}, \pm 1$$

- Plot

$\cup$ $\quad \cap$ $\quad \cup$

$[-\sqrt{2}$ $\quad -1$ $\quad 1$ $\quad \sqrt{2}]$

sign $f''$ $\quad +$ $\quad -$ $\quad +$

CU: $[-\sqrt{2}, -1) \cup (1, \sqrt{2}]$

CD: $(-1, 1)$

Inflection:

$\boxed{x = \pm 1 \quad \text{concavity changes}}$

(10) $f(x) = \dfrac{x^3}{3x^2+1}$

Derivatives

$$f'(x) = \frac{(3x^2+1)(3x^2) - x^3(6x)}{(3x^2+1)^2} = \frac{3x^4+3x^2}{(3x^2+1)^2}$$

$$f''(x) = \frac{(3x^2+1)^2(12x^3+6x) - (3x^4+3x^2)\left[2(3x^2+1)(6x)\right]}{(3x^2+1)^4} = \frac{-6x(x^2-1)}{(3x^2+1)^3}$$

Inc/Dec:

- Crit #'s of f ($f'=0$ on DNE)

$$f'(x) = \frac{3x^4+3x^2}{(3x^2+1)^2} \;\longrightarrow\; x=0$$

- Plot

$\nearrow \quad \nearrow$

sign $f'$ $\quad + \quad 0 \quad +$

$$\boxed{\begin{array}{l} I: (-\infty, \infty) \\ D: \text{none} \end{array}}$$

Extrema:

$x=0$

$f''(0)=0$ SDT inconclusive

$\nearrow \quad \nearrow$ at $0$

$\boxed{\text{FDT no extremum}}$

Concavity:

- Crit #'s of $f'$ ($f''=0$ or DNE)

$$f''(x) = \frac{-6x(x^2-1)}{(3x^2+1)^3} \;\longrightarrow\; x = \pm 1, 0$$

- Plot

$\cup \quad \cap \quad \cup \quad \cap$

sign $f''$ $\quad + \quad -1 \quad - \quad 0 \quad + \quad 1 \quad -$

$$\boxed{\begin{array}{l} CU: (-\infty,-1) \cup (0,1) \\ CD: (-1,0) \cup (1,\infty) \end{array}}$$

Inflection:

| | |
|---|---|
| $x = \pm 1$ | concavity changes |
| $x = 0$ | concavity changes |

(11) $f(x) = x^2 \ln x$ Domain: $(0, \infty)$

Derivatives:

$$f'(x) = 2x \ln x + x^2\left(\frac{1}{x}\right) = x(2\ln x + 1)$$

$$f''(x) = (2\ln x + 1) + x\left(\frac{2}{x}\right) = 2\ln x + 3$$

Inc/Dec

- Crit #'s of f ($f' = 0$ or DNE)

$$f'(x) = x(2\ln x + 1) = 0$$

$x = \cancel{0}$ not in domain

$\ln x = -\frac{1}{2}$

$x = e^{-1/2}$

- Plot

$\searrow \quad \nearrow$

sign $f'$ $\quad 0 \quad - \quad e^{-1/2} \quad +$

$$\boxed{\begin{array}{l} I: (e^{-1/2}, \infty) \\ D: (0, e^{-1/2}) \end{array}}$$

Extrema:

$x = e^{-1/2}$

$f''(e^{-1/2}) = +$ $\quad \boxed{\text{SDT min}}$

Concavity:

- Crit #'s of $f'$ ($f'' = 0$ or DNE)

$f''(x) = 2\ln x + 3 \quad \to \quad x = e^{-3/2}$

- Plot

$\cap \quad \cup$

sign $f''$ $\quad 0 \quad - \quad e^{-3/2} \quad +$

$$\boxed{\begin{array}{l} CU: (e^{-3/2}, \infty) \\ CD: (0, e^{-3/2}) \end{array}}$$

Inflection:

$\boxed{x = e^{-3/2} \quad \text{concavity changes}}$

(12) $f(x) = \cos x - 9x, \quad (0, 4\pi)$

Derivatives:

$f'(x) = -\sin x - 9$

$f''(x) = -\cos x$

Inc / Dec:

- Crit #'s of f ($f' = 0$ or DNE)

$f'(x) = -\sin x - 9 = 0 \qquad \phi$

- Plot

sign $f'$ $\quad -$

I: none

D: $(-\infty, \infty)$

Extrema:

$\boxed{\text{none}}$

Concavity

- Crit #'s of $f'$ ($f'' = 0$ or DNE)

$f''(x) = -\cos x \quad \longrightarrow \quad x = \frac{\pi}{2} + 2\pi n, \; n \in \mathbb{Z}$

- Plot

$\cap \quad \cup \quad \cap \quad \cup \quad \cap$

sign $f''$ $\quad - \quad \frac{\pi}{2} \quad + \quad \frac{3\pi}{2} \quad - \quad \frac{5\pi}{2} \quad + \quad \frac{7\pi}{2} \quad -$

Over $(0, 4\pi)$,

CU: $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{5\pi}{2}, \frac{7\pi}{2}\right)$

CD: $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \left(\frac{7\pi}{2}, 4\pi\right)$

Inflection

$\boxed{x = \frac{\pi}{2} + 2\pi n, \; n \in \mathbb{Z}}$

(13) $f(x) = \ln\sqrt{x^2+4} \quad = \frac{1}{2}\ln(x^2+4)$

Derivatives

$$f'(x) = \frac{x}{x^2+4}$$

$$f''(x) = \frac{(x^2+4)(1) - x(2x)}{(x^2+4)^2} = \frac{-x^2+4}{(x^2+4)^2}$$

Inc/Dec:

- Crit #'s of f ($f'=0$ or DNE)

$$f'(x) = \frac{x}{x^2+4} \quad \rightarrow \quad x = 0$$

- Plot

$\searrow \qquad \nearrow$

sign $f'$ $\quad - \quad 0 \quad +$

$\boxed{\begin{array}{l} I: (0,\infty) \\ D: (-\infty, 0) \end{array}}$

Extrema:

$x=0$

$f''(0) = +$ $\quad \boxed{\text{SDT min}}$

Concavity

- Crit #'s of $f'$ ($f''=0$ or DNE)

$$f''(x) = \frac{-x^2+4}{(x^2+4)^2} \quad \rightarrow \quad x = \pm 2$$

- Plot

$\cap \qquad \cup \qquad \cap$

sign $f''$ $\quad - \quad -2 \quad + \quad 2 \quad -$

$\boxed{\begin{array}{l} CU: (-2,2) \\ CD: (-\infty,-2)\cup(2,\infty) \end{array}}$

Inflection:

$\boxed{x = \pm 2}$

(14) $f(x) = xe^{-7x}$

Derivatives

$$f'(x) = e^{-7x} + x\left(-7e^{-7x}\right) = (1-7x)e^{-7x}$$

$$f''(x) = -7e^{-7x} + (1-7x)\left(-7e^{-7x}\right) = (49x-14)e^{-7x}$$

Inc / Dec:

– Crit #'s of F ($f'=0$ or DNE)

$$f'(x) = (1-7x)e^{-7x} \quad \to \quad x = \frac{1}{7}$$

– Plot

Sign $f'$   $+$   $\frac{1}{7}$   $-$

$$\boxed{\begin{array}{l} I: \left(-\infty, \frac{1}{7}\right) \\ D: \left(\frac{1}{7}, \infty\right) \end{array}}$$

Extrema

$x = \frac{1}{7}$

$f''\left(\frac{1}{7}\right) = -$   $\boxed{\text{SDT max}}$

Concavity

– Crit #'s of $f'$ ($f''=0$ or DNE)

$$f''(x) = (49x-14)e^{-7x} \quad \to \quad x = \frac{2}{7}$$

– Plot

$\cap$   $\cup$

sign $f''$   $-$   $\frac{2}{7}$   $+$

$$\boxed{\begin{array}{l} CU: \left(\frac{2}{7}, \infty\right) \\ CD: \left(-\infty, \frac{2}{7}\right) \end{array}}$$

Inflection:

$$\boxed{x = \frac{2}{7}}$$