19 Monday, October 16

Absolute Extrema on an Interval

Definition (Absolute Extrema). Let f be defined on an interval I containing c.

(i) $f(c)$ is the **minimum of f on I** when $f(c) \le f(x)$ for all x in I.

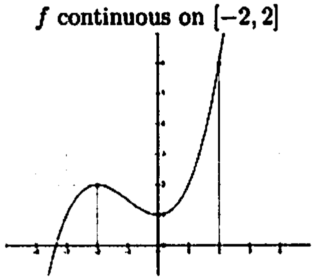
(ii) $f(c)$ is the **maximum of f on I** when $f(c) \ge f(x)$ for all x in I.

The maximum and minimum values are the **extreme values**, or **extrema**. In particular, the maximum and minimum of f on I are the **absolute maximum** and **absolute minimum** of f on I (**global** is also used for absolute). If extrema occur at endpoints, then they are called an **endpoint extrema**.

Theorem (Extreme Value Theorem). If f is continuous on a closed interval $[a, b]$, then f attains both a maximum and minimum on the interval.

Note. The continuity condition on $[a, b]$ is necessary for the EVT to hold.

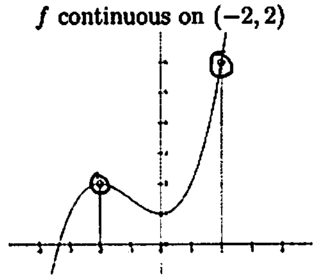
f continuous on $[-2, 2]$



min at $x=0$

max at $x=2$

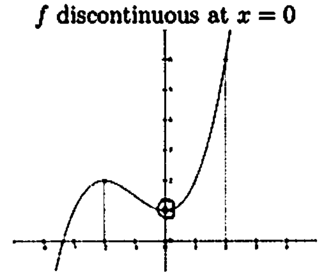
f continuous on $(-2, 2)$



min at $x=0$

no max

f discontinuous at $x = 0$



no min

max at $x=2$

To find absolute extrema of a continuous function f on $[a, b]$:

(1) Find the critical numbers of f that lie in $[a, b]$

(2) Evaluate f at each critical number.

(3) Evaluate f at the endpoints a and b.

(4) The max/min of these numbers is the absolute extrema.

Example. Find the absolute extrema of each function on the given interval.

(1) $f(x) = 4 - x^2$ on $[-3, 1]$

- Find crit #'s in $[-3,1]$

  $f'(x) = -2x = 0 \quad \rightarrow \quad x = 0$

- Evaluate f at crit #'s and endpoints

  $f(-3) = 4 - 9 = -5$ min

  $f(0) = 4 - 0 = 4$ max

  $f(1) = 4 - 1 = 3$

(2) $f(x) = x^3 - \frac{3}{2}x^2$ on $[-1, 2]$

- Find crit #'s in $[-1, 2]$

  $f'(x) = 3x^2 - 3x = 0 \quad \rightarrow \quad x = 0, 1$

- Evaluate f at crit #'s and endpoints

  $f(-1) = -1 - \frac{3}{2} = -\frac{5}{2}$ min

  $f(0) = 0 = 0$

  $f(1) = 1 - \frac{3}{2} = -\frac{1}{2}$

  $f(2) = 8 - 6 = 2$ max

(3) $f(x) = -\frac{1}{x^2}$ on $[1, 2]$

- Find crit #'s in $[1,2]$

$f'(x) = \frac{2}{x^3}$ → $x = \cancel{0}$ not in interval

- Evaluate f at crit #'s and endpoints

$f(1) = -1$ min

$f(2) = -\frac{1}{4}$ max

(4) $f(x) = x + \frac{1}{x}$ on $\left[\frac{1}{2}, 3\right]$

- Find crit #'s in $[\frac{1}{2}, 3]$

not in interval

$f'(x) = 1 - \frac{1}{x^2}$ → $x = \cancel{-1}, \cancel{0}, 1$

- Evaluate f at crit #'s and endpoints

$f(\frac{1}{2}) = \frac{1}{2} + 2 = \frac{5}{2}$

$f(1) = 1 + 1 = 2$ min

$f(3) = 3 + \frac{1}{3} = \frac{10}{3}$ max

(5) $f(x) = \frac{3}{x-5}$ on $[0, 2]$

- Find crit #'s in $[0,2]$

$f'(x) = \frac{-3}{(x-5)^2} \rightarrow x = \cancel{5}$ not in interval

- Evaluate f at crit #'s and endpoints

$f(0) = -\frac{3}{5}$ max

$f(2) = -1$ min

(6) $f(x) = \frac{x^2}{x+1}$ on $\left[-\frac{1}{2}, 1\right]$

- Find crit #'s in $[-\frac{1}{2}, 1]$

not in interval

$f'(x) = \frac{(x+1)(2x) - x^2(1)}{(x+1)^2} = \frac{x^2+2x}{(x+1)^2} \rightarrow x = \cancel{-2}, \cancel{-1}, 0$

- Evaluate f at crit #'s and endpoints.

$f(-\frac{1}{2}) = \frac{1/4}{1/2} = \frac{1}{2}$ max

$f(0) = 0$ min

$f(1) = \frac{1}{2}$ max

(7) $f(x) = \sqrt{4 - x^2}$ on $[-1, 2]$

- Find crit #'s in $[-1,2]$

not in interval

$f'(x) = \dfrac{-x}{\sqrt{4-x^2}} \quad \rightarrow \quad x = \cancel{-2}, 0, 2$

- Evaluate f at crit #'s and endpoints

$f(-1) = \sqrt{3}$

$f(0) = 2$ max

$f(2) = 0$ min

(8) $f(x) = 3x^{2/3} - 2x$ on $[-1, 1]$

- Find crit #'s in $[-1,1]$

$f'(x) = 2x^{-1/3} - 2 \quad \rightarrow \quad x = 0, 1$

- Evaluate f at crit #'s and endpoints

$f(-1) = 3 + 2 = 5$ max

$f(0) = 0 = 0$ min

$f(1) = 3 - 2 = 1$

(9) $f(x) = \sec x$ on $\left[-\frac{\pi}{3}, \frac{\pi}{6}\right]$

- Find crit #'s in $\left[-\frac{\pi}{3}, \frac{\pi}{6}\right]$

$f'(x) = \sec x \tan x = \frac{\sin x}{\cos^2 x} \longrightarrow \begin{array}{l} x = \pi n, \ n \in \mathbb{Z} \\ x = \frac{\pi}{2} + \pi n, \ n \in \mathbb{Z} \end{array}$ but only $x=0$ in $\left[-\frac{\pi}{3}, \frac{\pi}{6}\right]$

- Evaluate f at crit #'s and endpoints

$f(-\frac{\pi}{3}) = 2$ max

$f(0) = 1$ min

$f(\frac{\pi}{6}) = \frac{2}{\sqrt{3}}$

(10) $f(x) = x^2 e^x$ on $[-3, 1]$

- Find crit #'s in $[-3, 1]$

$f'(x) = x^2 e^x + 2xe^x = (x^2+2x)e^x \longrightarrow x = -2, 0$

- Evaluate f at crit #'s and endpoints

$f(-3) = 9e^{-3}$ $\qquad \frac{9}{e^3} < \frac{9}{2^3} < 2 < e$

$f(-2) = 4e^{-2}$ $\qquad \frac{4}{e^2} < \frac{4}{2^2} = 1 < e$

$f(0) = 0$ min

$f(1) = e$ max

(11) $f(x) = 5e^x \sin x$ on $[0, \pi]$

- Find crit #'s in $[0,\pi]$

$f'(x) = 5e^x \sin x + 5e^x \cos x = 5e^x(\sin x + \cos x) = 0$

$5e^x = 0$

ϕ

$\sin x + \cos x = 0$

$\tan x = -1$

$x = -\frac{\pi}{4} + \pi n, \quad n \in \mathbb{Z}$

but only $x = \frac{3\pi}{4}$ is in $[0,\pi]$.

- Evaluate f at crit #'s and endpoints

$f(0) = 0$ min

$f(\frac{3\pi}{4}) = 5e^{3\pi/4}\frac{\sqrt{2}}{2}$ max

$f(\pi) = 0$ min

(12) $f(x) = x^2 - 8 \ln x$ on $[1, 6]$

- Find crit #'s in $[1,6]$

not in interval

$f'(x) = 2x - \frac{8}{x} = \frac{2x^2-8}{x} \rightarrow x = \cancel{-2}, \cancel{0}, 2$

- Evaluate f at crit #'s and endpoints

$f(1) = 1$

$f(2) = 4 - 8\ln 2$ min

$f(6) = 36 - 8\ln 6$ max

(13) ~~$f(x) = x\ln(x+3)$ on $[0,3]$~~

(14) $f(x) = \dfrac{\ln x}{8x}$ on $[1,4]$

- Find crit #'s in [1, 4].

$$f'(x) = \frac{8x\left(\frac{1}{x}\right) - 8\ln x}{64x^2} = \frac{8 - 8\ln x}{64x^2} \longrightarrow x = \cancel{0}, e$$

not in interval

- Evaluate f at crit #'s and endpoints

$f(1) = 0$ min

$f(e) = \dfrac{1}{8e}$ max

$f(4) = \dfrac{\ln 4}{32}$