20 Friday, October 20

Limits at Infinity

Definition (Limits at Infinity). If the values of the function $f(x)$ approach the number L as x grows without bound, then

$$\lim_{x\to\infty} f(x) = L$$

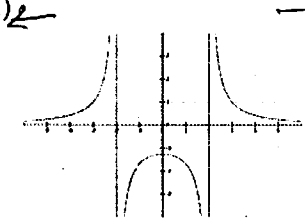
and similarly

$$\lim_{x\to-\infty} f(x) = M.$$

These limiting values, if the exist, create horizontal asymptotes.

Example. Find the horizontal asymptotes of the following graphs.

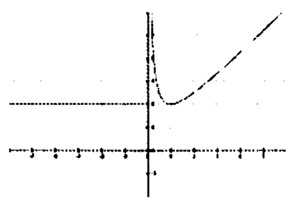
(1)



$y = 0$

$\lim_{x\to\pm\infty} f(x) = 0$

(2)

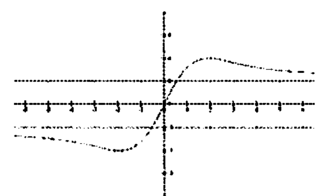


$y = 2.$

$\lim_{x\to-\infty} f(x) = 2.$

$\lim_{x\to\infty} f(x) = +\infty$

(3)



$y = \pm 1.$

$\lim_{x\to\infty} f(x) = 1$

$\lim_{x\to-\infty} f(x) = -1.$

Example. Use a calculator to complete the table and estimate the limit as x approaches infinity.

$$f(x) = \frac{10x+7}{5x-3}$$

x	1	10	10^2	10^3	10^4	10^5	10^6
$f(x)$	8.5	2.28	2.03	2.003	2.000	2.000	2.000

$\frac{17}{2}$ $\quad$ $\frac{107}{47}$

$\lim_{x\to\infty} f(x) = 2.$

Theorem (Reciprocal Limit Rule). If A is a real number, $k > 0$, and x^k is defined for all x, then

$$\lim_{x\to\pm\infty} \frac{A}{x^k} = 0.$$

To find limits at infinity:

(1) Divide each term in the numerator and denominator by the highest power of x in the numerator and denominator.

(2) Apply limit rules.

Example. Find each limit.

(1) $\lim_{x\to\infty}\left(5+\frac{8}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{8}{x} = 5+0=5.$

(2) $\lim_{x\to\infty}\frac{5-7x}{5x^3-9}$

Highest power $= x^3$

$$= \lim_{x\to\infty}\frac{5-7x}{5x^3-9}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to\infty}\frac{\frac{5}{x^3}-\frac{7}{x^2}}{5-\frac{9}{x^3}} = \frac{0-0}{5-0} = 0.$$

(3) $\lim_{x\to\infty}\frac{5-7x}{5x-9}$

Highest power $= x$

$$= \lim_{x\to\infty}\frac{5-7x}{5x-9}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\frac{5}{x}-7}{5-\frac{9}{x}} = \frac{0-7}{5-0} = -\frac{7}{5}$$

(4) $\lim_{x\to\infty}\frac{5-7x^2}{5x-9}$

Highest power $= x^2$

$$= \lim_{x\to\infty}\frac{5-7x^2}{5x-9}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty}\frac{\frac{5}{x^2}-7}{\frac{5}{x}-\frac{9}{x^2}} = \frac{-7}{0}$$

When x is very large,

$$\left.\begin{array}{ll} 5-7x^2 & \text{is negative} \\ 5x-9 & \text{is positive}\end{array}\right\} \longrightarrow \frac{5-7x^2}{5x-9} \text{ is negative}$$

$$\lim_{x\to\infty}\frac{5-7x^2}{5x-9} = -\infty \qquad \lim_{x\to-\infty}\frac{5-7x^2}{5x-9} = +\infty$$

(5) $\lim_{x\to\infty} \frac{x^2-2x+5}{2x^2+5x+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1-\frac{2}{x}+\frac{5}{x^2}}{2+\frac{5}{x}+\frac{1}{x^2}} = \frac{1}{2}.$

(6) $\lim_{x\to-\infty} \frac{2x+1}{3x^2+2x-7} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to-\infty} \frac{\frac{2}{x}+\frac{1}{x^2}}{3+\frac{2}{x}-\frac{7}{x^2}} = \frac{0+0}{3+0-0} = 0$

(7) $\lim_{x\to-\infty} \frac{9x^3+7}{18x^3-6x^2+4} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}} = \lim_{x\to-\infty} \frac{9+\frac{7}{x^3}}{18-\frac{6}{x}+\frac{4}{x^3}} = \frac{1}{2}.$

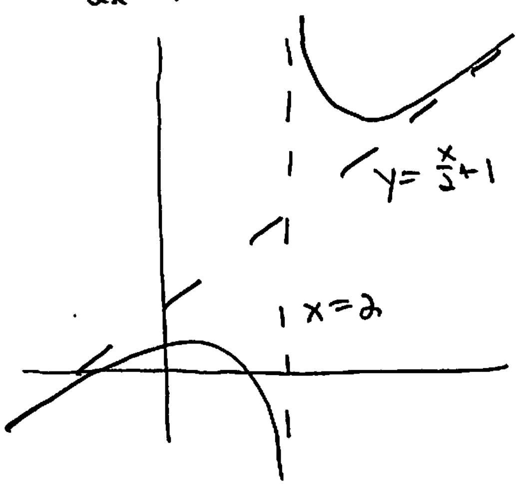
Definition. A rational function $f(x) = p(x)/q(x)$ has a **dominant term** $g(x)$ as $x \to \infty$ (or $x \to -\infty$) if it can be written such that

$$f(x) = g(x) + \frac{r(x)}{q(x)}, \qquad \text{where} \qquad \lim_{x\to\infty} \frac{r(x)}{q(x)} = 0.$$

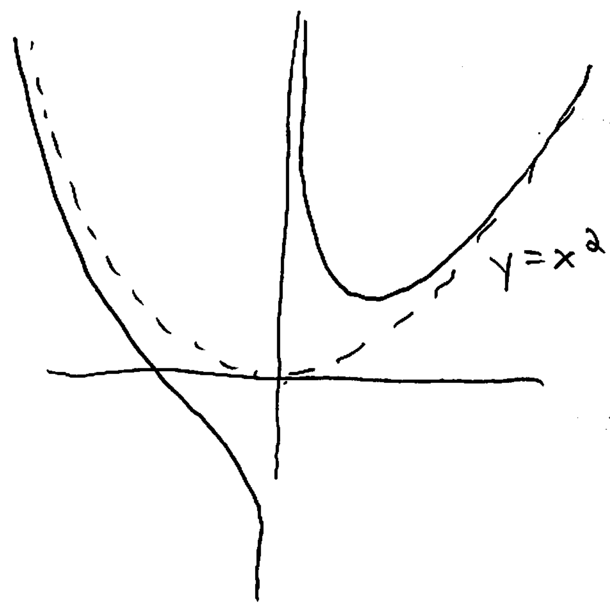
Dominant terms are functions such that when x is very large, then $f(x) \approx g(x)$ so that $g(x)$ *dominates* the $r(x)/q(x)$ term. In particular, when p and q are polynomials, f has a dominant term when $\deg p \geq \deg q$, and when $\deg p = \deg q + 1$, then f has $g(x)$ for a **slant asymptote**.

Ex:

$$f(x) = \frac{x^2-3}{2x-4} = \frac{x}{2} + 1 + \frac{1}{2x-4}$$



$$f(x) = \frac{x^3+1}{x} = x^2 + \frac{1}{x}$$



Theorem (Rational Function Theorem). Let

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} \quad \text{with } \deg p = n, \deg q = m$$

be a rational function. Then:

(1) $f(x)$ has a vertical asymptote $x = c$ if $q(c) = 0$ but $p(c) \neq 0$. If both $p(c) = q(c) = 0$, then factor $f(x)$ and perform any cancellations. Check $p(c)$ and $q(c)$ afterwards

(2) $f(x)$ has the following possibilities for a horizontal asymptote:

(a) if $n > m$, then f has no horizontal asymptotes, and if $n = m + 1$, then f has a slant asymptote.

(b) if $n < m$, then $y = 0$ is a horizontal asymptote.

(c) if $n = m$, then $y = \frac{a_n}{b_n}$ is a horizontal asymptote.

$$\frac{x^2-1}{x-1}$$

and slant

Example. Find the horizontal and vertical asymptotes of the following functions.

(1) $f(x) = \frac{1}{x-1}$ deg $= 0$ deg $= 1$

$x - 1 = 0$

$x = 1$

HA: $y = 0$

VA: $x = 1$

(2) $f(x) = \frac{x+3}{x-2}$ deg $= 1$ deg $= 1$

HA: $y = \frac{1}{1} = 1$

VA: $x = 2$

(3) $f(x) = \frac{x^2+x-6}{x^2-16}$

HA: $y = \frac{1}{1} = 1$

VA: $x = \pm 4$

(4) $f(x) = \frac{2x^2+x-1}{x^2-1} = \frac{(2x-1)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}}$

HA: $y = \frac{2}{1} = 2$

$x^2 - 1 = 0$

$x = \pm 1$

VA: $x = 1$

Hole: $x = -1$

(5) $f(x) = \frac{4x}{x^2+4}$ deg = 1, deg = 2

HA: $y=0$

VA: none

$x^2+4=0$

$x^2=-4$

(6) $f(x) = \frac{2x^2}{x+1} = 2x-2+\frac{2}{x+1}$

Use long division

$$\begin{array}{r l} & 2x - 2 \quad r\ 2. \\ x+1 \,\big) & 2x^2 + 0x + 0 \\ (-) & 2x^2 + 2x \\ \hline & -2x + 0 \\ (-) & -2x - 2 \\ \hline & 2 \end{array}$$

HA: none

VA: $x=-1$.

SA: $y=2x-2$

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(7) $f(x) = \frac{x^2-4}{x-1}$

$$\begin{array}{r l} & x + 1 \\ x-1 \,\big) & x^2 + 0x - 4 \\ (-) & x^2 - x \\ \hline & x - 4 \\ (-) & x - 1 \\ \hline & -3 \end{array}$$

HA: none

VA: $x=1$.

SA: $y=x+1$.

(8) $f(x) = \frac{x^4+1}{x^2}$

HA: none

VA: $x=0$

SA: none

(9) $f(x) = \dfrac{x^3 + x - 2}{x - x^2}$

(10) $f(x) = \dfrac{x^3 + 3x + 1}{x^2 + x - 2}$

HA: none

VA: $x = 1, -2.$

SA: $y = x - 1$

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad x - 1 \\
x^2 + x - 2 & x^3 + 0x^2 + 3x + 1 \\
 & (-)\ x^3 + x^2 - 2x \quad \downarrow \\
\hline
 & \quad -x^2 + 5x + 1
\end{array}
$$

Example. Find the following limits.

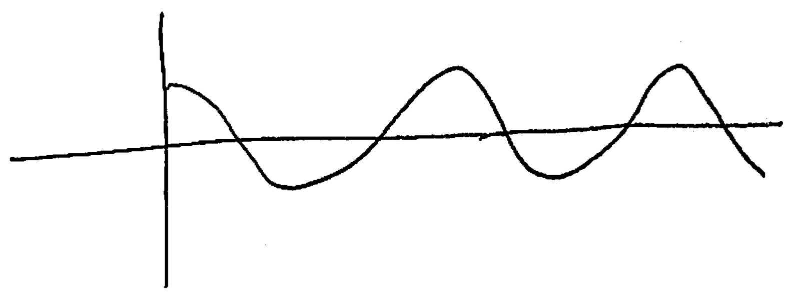
(1) $\lim_{x \to \infty} \frac{5 - 2x^{3/2}}{3x^2 - 4}$ deg = $\frac{3}{2}$ deg = 2 $= 0.$

(2) $\lim_{x \to -\infty} \frac{x}{\sqrt{x^2 - x}}$

(3) $\lim_{x \to -\infty} \frac{\sqrt{x^4 - 1}}{x^3 - 1}$ deg = 2 deg = 3 $= 0$

(4) $\lim_{x \to -\infty} \dfrac{2x}{(x^6 - 1)^{1/3}}$ deg = 1 deg = $6 \cdot \frac{1}{3} = 2$. $= 0$

(5) $\lim_{x \to \infty} \cos x$ = DNE



(6) $\lim_{x \to \infty} \sin \dfrac{1}{x}$ $= 0$

(7) $\lim\limits_{x \to \infty} \frac{1}{e^x + 1} = 0$

(8) $\lim\limits_{x \to -\infty} \frac{1}{e^x + 1} = \frac{1}{0+1} = 1$