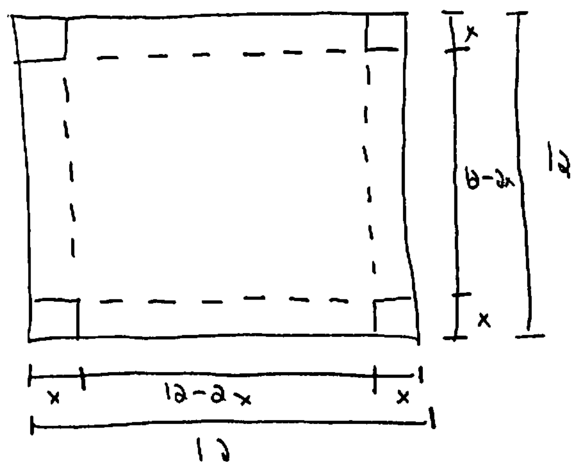
23 Wednesday, October 25 to Monday, October 30

Optimization Problems

(1) An open-top box is to be made by cutting small congruent squares from the corners of a 12-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



Volume of box:

$$V(x) = \underbrace{(12-2x)}_{\text{length}}\underbrace{(12-2x)}_{\text{width}}\underbrace{x}_{\text{height}} = 4(6-x)^2 x \quad \text{where } 0 \le x \le 6$$

Maximize $V(x)$ over $[0,6]$.

$$V'(x) = 8(6-x)(-1)x + 4(6-x)^2(1) = 0$$

$$4(6-x)[-2x + (6-x)] = 0$$

$$4(6-x)(6-3x) = 0$$

$$x = 2, 6$$

Evaluate $V(x)$ at crit #'s and endpoints

$V(0) = 0$

$V(2) = 4(4)^2(2) = 128$

$V(6) = 0$

$\boxed{\text{So } x=2 \text{ and the dimensions are } 8\times 8\times 2.}$

(2) Find two positive numbers whose sum is 20 and whose product is as large as possible.

$$\begin{cases} x+y=20 \\ xy=p \end{cases} \longrightarrow \quad \begin{array}{l} y=20-x \\ p(x)=x(20-x). \end{array}$$

Maximize $p(x)=x(20-x)$ over $0 \le x \le 20$

$p'(x)=20-2x=0 \quad \rightarrow \quad x=10$

Evaluate p at crit #'s and endpoints:

$p(0)=0$

$\boxed{p(10)=100 \qquad x, y=10}$

$p(20)=0$

(3) Find two positive numbers whose product is 768 and sum of the first plus three times the second is a minimum.

$$\begin{cases} xy=768 \\ x+3y=f \end{cases} \longrightarrow \quad \begin{array}{l} y=\dfrac{768}{x} \\ f(x)=x+\dfrac{768(3)}{x} \end{array}$$

Minimize $f(x)$ over $(0,\infty)$

$f'(x)=1-\dfrac{768(3)}{x^2}=0$

not in interval

crit #'s $= -\cancel{48}, \cancel{0}, 48$

$x^2=768(3)$

$x=\pm 48$

Is $x=48$ a max/min/neither? Use FDT

$\searrow \quad \nearrow$

sign f' $\quad 0 \quad 48$

$- \qquad +$

FDT min

$\boxed{\text{So } x=48, \; y=\dfrac{768}{48}=16}$

(4) **Find the length and width of a rectangle that has a perimeter of 124 meters and a maximum area.**

$$\begin{cases} 2x+2y=124 \\ xy=A \end{cases} \longrightarrow \begin{array}{l} y=62-x \\ A(x)=x(62-x) \end{array}$$

Maximize $A(x)$ over $[0,62]$.

$$A'(x)=62-2x=0 \longrightarrow x=31$$

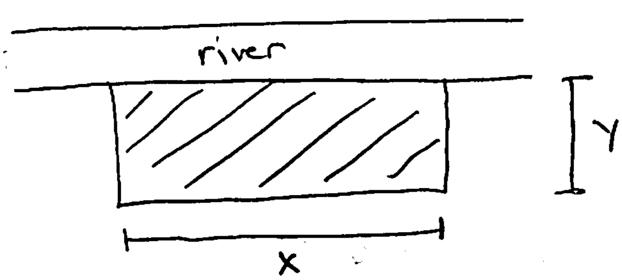
Evaluate A at crit #'s and endpoints:

$$A(0)=0$$

$$\boxed{A(31)=961 \longrightarrow x=31=y}$$

$$A(62)=0$$

(5) **A farmer plans to enclose a rectangular pasture adjacent to a river (see figure). The pasture must contain 80,000 square meters in order to provide enough grass for the herd. No fencing is needed along the river. What dimensions will require the least amount of fencing?**



$$\begin{cases} xy=80{,}000 \\ x+2y=f \end{cases} \longrightarrow \begin{array}{l} y=\dfrac{80{,}000}{x} \\ f(x)=x+\dfrac{160{,}000}{x} \end{array}$$

Minimize $f(x)$ over $(0,\infty)$.

not in interval

$$f'(x)=1-\frac{160{,}000}{x^2}=0 \qquad \text{crit \#'s} = \cancel{-400},\ 400$$

$$x=\pm 400$$

Is $x=400$ a max / min / neither? Use FDT

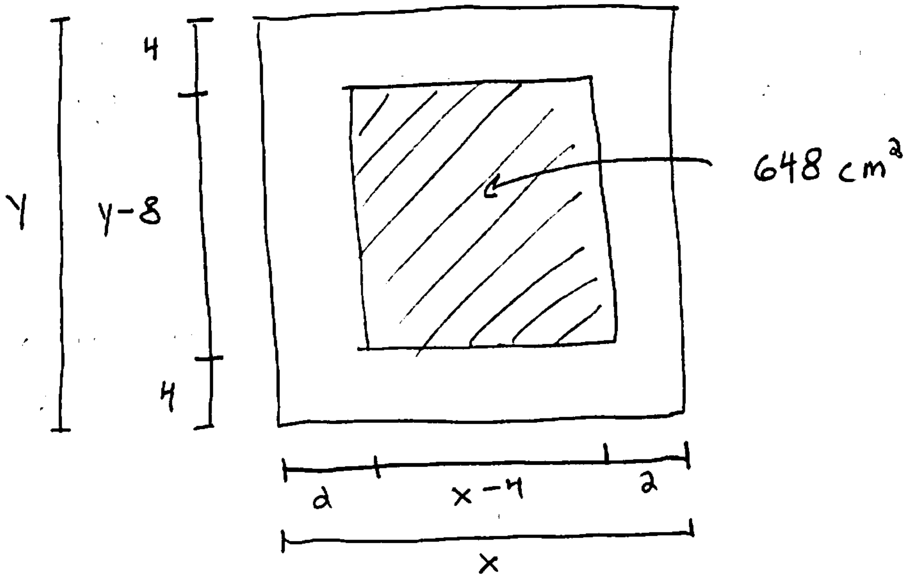
$\searrow$ $\nearrow$

Sign f': 0 $-$ 400 $+$

FDT min

$$\boxed{\text{So } x=400,\ y=\frac{80{,}000}{400}=200}$$

(6) You are asked to create a rectangular poster containing 648 cm^2 of print surrounded by margins of 2 cm on the sides and 4 cm on top and bottom that uses minimal paper.



$$\begin{cases} (x-4)(y-8) = 648 \\ xy = A \end{cases} \quad \to \quad \begin{aligned} y - 8 &= \frac{648}{x-4} \\ y &= \frac{648}{x-4} + 8 \\ A &= x\left(8 + \frac{648}{x-4}\right) \end{aligned}$$

Minimize A over $(4, \infty)$

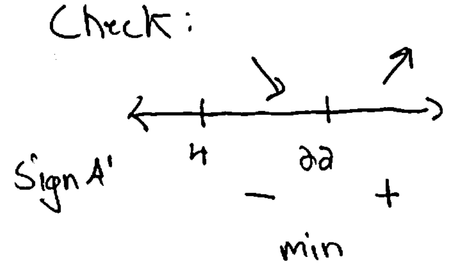
$$A'(x) = 8 + \frac{648(x-4) - 648x(1)}{(x-4)^2} = 8 - \frac{4(648)}{(x-4)^2} = 0$$

$$(x-4)^2 = 324$$

$$x - 4 = \pm 18$$

$$x = \cancel{-14}, 22$$

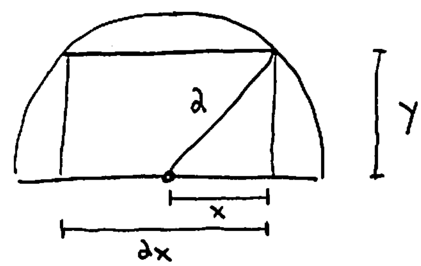
Check:



$$\boxed{\text{So } x = 22, \; y = 8 + \frac{648}{18} = 44}$$

(7) **A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?**

$$\begin{cases} x^2+y^2=4 \\ 2xy=A \end{cases} \rightarrow \begin{array}{l} y=\sqrt{4-x^2} \\ A=2x\sqrt{4-x^2} \end{array}$$



Maximize A over $[0,2]$

$$A'(x)=2\sqrt{4-x^2}+2x\left[\tfrac{1}{2}(4-x^2)^{-1/2}(-2x)\right]=\frac{2(4-x^2)-2x^2}{(4-x^2)^{1/2}}=\frac{8-4x^2}{(4-x^2)^{1/2}}$$

$x=\pm 2, \pm\sqrt{2}.$

Evaluate A at crit #'s and endpoints.

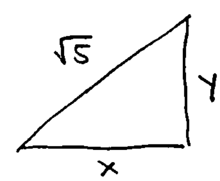
$A(0)=0$

$\boxed{A(\sqrt{2})=2\sqrt{2}\sqrt{2}=4 \qquad x=\sqrt{2},\ y=\sqrt{2}}$

$A(2)=0$

(8) **Find the largest possible value of $s=2x+y$ if x and y are side lengths in a right triangle whose hypotenuse is $\sqrt{5}$ units long.**

$$\begin{cases} x^2+y^2=5 \\ 2x+y=s \end{cases} \rightarrow \begin{array}{l} y=\sqrt{5-x^2} \\ s=2x+\sqrt{5-x^2} \end{array}$$



Maximize s over $[0,\sqrt{5}]$

$$s'(x)=2+\tfrac{1}{2}(5-x^2)^{-1/2}(-2x)=\frac{2(5-x^2)^{1/2}-x}{(5-x^2)^{1/2}}=0$$

$2(5-x^2)^{1/2}=x$

$4(5-x^2)=x^2$

$20=5x^2$

$x=\pm 2$

Evaluate s at crit #'s and endpoints

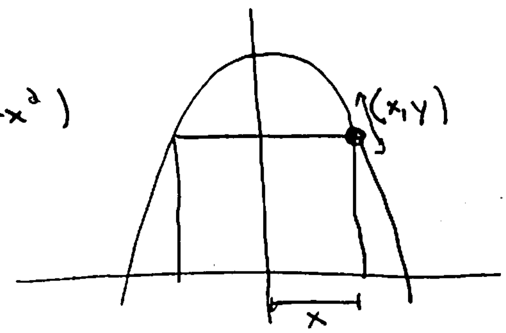
$s(0)=\sqrt{5}$

$\boxed{s(2)=4+1=5}$

$s(\sqrt{5})=2\sqrt{5}$

(9) A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have?

$$\begin{cases} y = 12 - x^2 \\ 2xy = A \end{cases} \rightarrow A = 2x(12 - x^2)$$



Maximize A over $[0, \sqrt{12}]$

$$A'(x) = 24 - 6x^2 = 0$$

$$x = \pm 2$$

Evaluate A at crit #'s and endpoints:

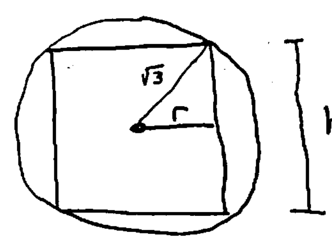
$A(0) = 0$

$\boxed{A(2) = 4(8) = 32}$

$A(\sqrt{12}) = 0$

(10) Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$. The volume of a cylinder is $V = \pi r^2 h$.

$$\begin{cases} r^2 + \left(\frac{h}{2}\right)^2 = 3 \\ \pi r^2 h = V \end{cases} \rightarrow \begin{matrix} h = \sqrt{12 - 4r^2} \\ V = \pi r^2 \sqrt{12 - 4r^2} \end{matrix}$$



Maximize V over $[0, \sqrt{3}]$

$$V'(r) = 2\pi r(12 - 4r^2)^{1/2} + \pi r^2\left[\tfrac{1}{2}(12 - 4r^2)^{-1/2}(-8r)\right]$$

$$= \frac{2\pi r(12 - 4r^2) - 4\pi r^3}{(12 - 4r^2)^{1/2}} = \frac{24\pi r - 12\pi r^3}{(12 - 4r^2)^{1/2}}$$

$$r = 0, \pm\sqrt{2}, \pm\sqrt{3}$$

Evaluate V at crit #'s and endpoints

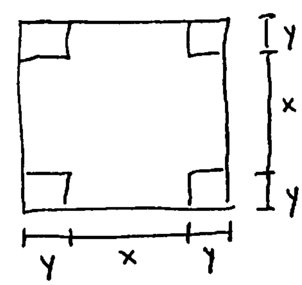
$V(0) = 0$

$\boxed{V(\sqrt{2}) = 2\pi\sqrt{12 - 8} = 4\pi}$

$V(\sqrt{3}) = 0$

(11) **An open-top box with a square base is to have a volume of 32 cubic feet. Find the dimensions of the box that can be made with the smallest amount of material.**

$$\begin{cases} x^2y = 32 \\ 4xy + x^2 = S \end{cases} \rightarrow \quad y = \frac{32}{x^2}, \quad S = x^2 + \frac{128}{x}$$



Minimize S over $(0, \infty)$

$$S'(x) = 2x - \frac{128}{x^2} = 0$$

$$x^3 = 64$$

$$x = 4$$

Check

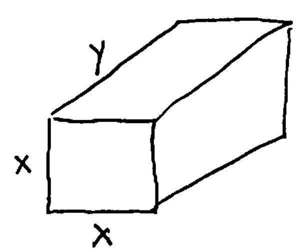
$\searrow \quad \nearrow$

sign S': 0 $\;-\;$ 4 $\;+\;$

$\boxed{\text{Dimensions: } 4 \times 4 \times 2}$

(12) **A rectangular package to be sent by a postal service can have a maximum combined length and girth (perimeter of a cross section) of 90 inches. Find the dimensions of the package of maximum volume that can be sent. (Assume the cross section is square.)**

$$\begin{cases} 4x + y = 90 \\ x^2y = V \end{cases} \rightarrow \quad y = 90 - 4x, \quad V = x^2(90 - 4x)$$



Maximize V over $[0, \frac{90}{4}]$

$$V'(x) = 180x - 12x^2 = 0$$

$$x = 0, 15$$

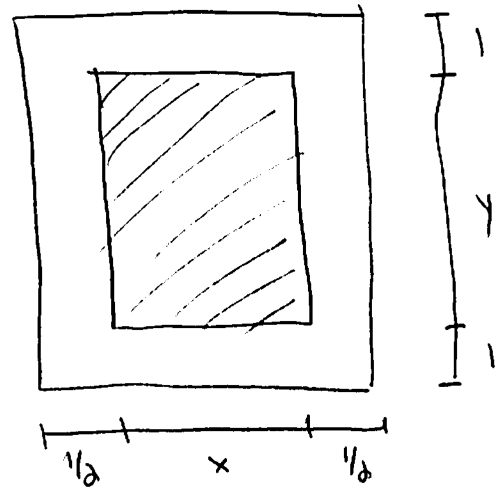
Evaluate V at crit #'s and endpoints

$V(0) = 0$

$\boxed{V(15) = 225(30) = 6750 \qquad \text{Dimensions: } 15 \times 15 \times 30}$

$V(\frac{90}{4}) = 0$

(13) A production editor at Weston Publishers decided that the pages of a book should have a 1-in. margin at the top and the bottom, and a 1/2-in. margin on each side of the page. She further stipulated that each page of the book should have an area of 32 in.^{2}. Determine the dimensions of the page that will result in the maximum printed area on the page.



$$\begin{cases} (y+2)(x+1) = 32 \\ xy = A \end{cases} \quad \rightarrow \quad y+2 = \frac{32}{x+1}$$

$$y = \frac{32}{x+1} - 2$$

$$A = x\left[\frac{32}{x+1} - 2\right]$$

Maximize A over $(0, \infty)$

$$A'(x) = \frac{32(x+1) - 32x(1)}{(x+1)^2} - 2 = 0$$

$$32 = 2(x+1)^2$$

$$16 = (x+1)^2$$

$$x+1 = \pm 4$$

$$x = \cancel{-5}, 3$$

Check

Sign A' 0   +   3   −

$\boxed{\text{Page Dimensions: } 4 \times 8}$

(14) The sum of the perimeters of a circle and a square is 12. Find the radius and side length of the square that produces a minimum total area.

$2\cdot 4\sqrt{5}$

$12\sqrt{20}$

$\sqrt{2880}$

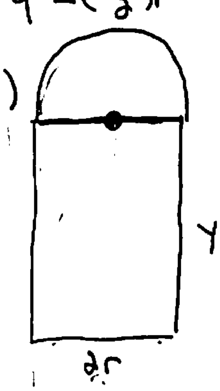
$\sqrt{144\times 20}$

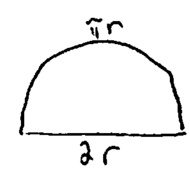
(15) A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions of a Norman window of maximum area if the total perimeter is 18 feet.

$$\begin{cases} 2r+2y+\pi r = 18 \\ 2ry + \frac{1}{2}\pi r^2 = A \end{cases} \longrightarrow \quad y = \frac{18-2r-\pi r}{2} = 9 - \left(\frac{2+\pi}{2}\right)r$$

$$A = \tfrac{1}{2}\pi r^2 + (18r - 2r^2 - \pi r^2)$$

$$= 18r - 2r^2 - \tfrac{1}{2}\pi r^2$$





Maximize A over $\left[0, \frac{18}{2+\pi}\right]$.

$$A'(r) = 18 - 4r - \pi r = 0$$

$$r = \frac{18}{4+\pi}$$

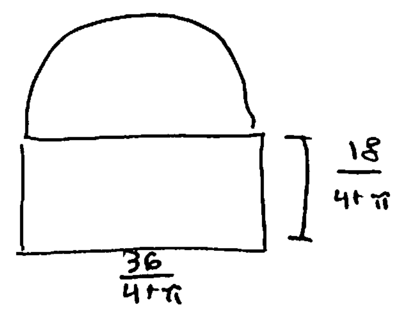
Check max

Sign A' $\quad + \quad \frac{18}{4+\pi} \quad -$

$$y = 9 - \left(\frac{2+\pi}{2}\right)\cdot\frac{18}{4+\pi}$$

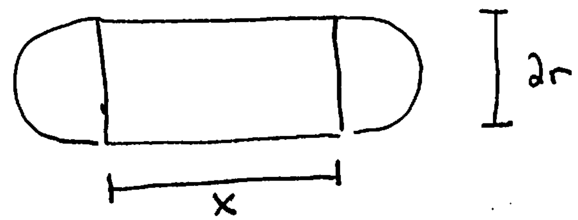
$$= \frac{9(4+\pi)}{4+\pi} - \frac{9(2+\pi)}{4+\pi}$$

$$= \frac{18}{4+\pi}$$



(16) A racetrack encloses a football field with ends that are semicircular. The length of the track is 1760 ft (1/3 mi). Find the dimensions of the rectangular field so that the area of the field is as large as possible, and compute the total area enclosed by the track.

$$\begin{cases} 2x + 2\pi r = 1760 \\ 2xr = A \end{cases}$$



$$x = \frac{1760 - 2\pi r}{2}$$

$$= 880 - \pi r$$

$$A = 2(880 - \pi r) r = 1760r - 2\pi r^2$$

Maximize A over $(0, \infty)$.

$$A'(r) = 1760 - 4\pi r = 0$$

$$r = \frac{1760}{4\pi} = \frac{440}{\pi}$$

$$x = 880 - \pi\left(\frac{440}{\pi}\right)$$

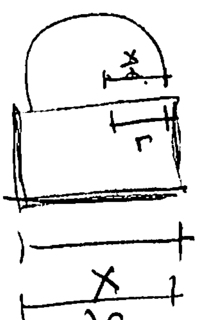
$$= 440$$

Dimensions: $440 \times \frac{880}{\pi}$

Total area enclosed by track

$$= 2xr + \pi r^2$$

$$= 2(440)\left(\frac{440}{\pi}\right) + \pi\left(\frac{440}{\pi}\right)^2$$



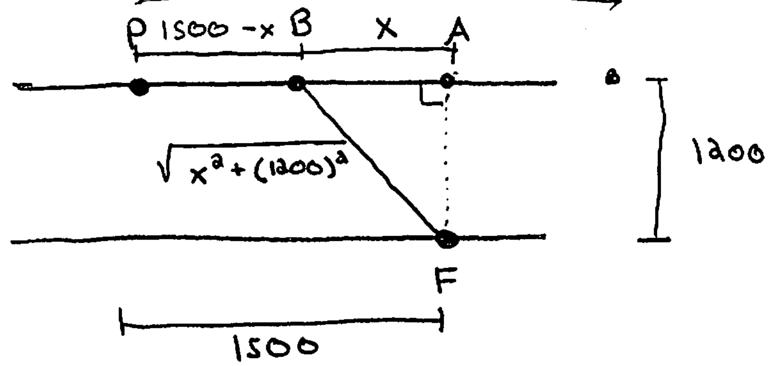
$x + 2y + \pi\left(\frac{x}{2}\right)$

$2r + 2y + \pi r$

$\frac{24}{\pi + 4}$

$x = \frac{48}{\pi + 4}$

(17) **A cable is to be run from a power plant on one side of a river 1,200 m wide to a factory on the other side 1,500 m downstream. The cable costs \$25/ m to lay underwater, and \$20/ m to lay on land. What is the most economical way to run the cable.**



The cost of the cable:

$$C(x) = \underbrace{(1500-x)(20)}_{\text{land cost}} + \underbrace{25\sqrt{x^2+(1200)^2}}_{\text{underwater cost}}$$

x should be between 0 and 1500 to prevent the cable from backtracking.

Minimize C over $[0, 1500]$

$$C'(x) = -20 + \frac{25}{\cancel{2}}\left(x^2+(1200)^2\right)^{-1/2}(\cancel{2}x) = 0$$

$$\frac{25x}{\sqrt{x^2+(1200)^2}} = 20$$

$$25x = 20\sqrt{x^2+(1200)^2}$$

$$5x = 4\sqrt{x^2+(1200)^2}$$

$$25x^2 = 16\left(x^2+(1200)^2\right)$$

$$25x^2 = 16x^2 + 16(1200)^2$$

$$9x^2 = 16(1200)^2$$

$$3x = 4(1200)$$

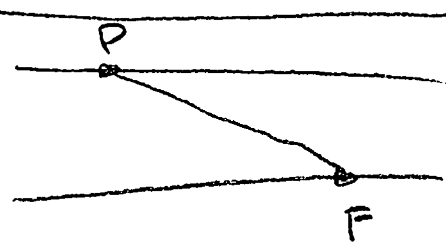
$$x = 4(400) = \cancel{1600}$$

not in interval

Evaluate C at crit #'s and endpoints:

$C(0) = 30{,}000 + 30{,}000 = 60{,}000$

$\boxed{C(1500) = 7500\sqrt{41} \approx 48{,}000}$



(18) Suppose now the fact is 2,000 m downstream. What is the most economical way to run the cable?

$$C(x) = (2000-x)(20) + 25\sqrt{x^2+(1200)^2}$$

Minimize C over $[0, 2000]$.

$$C'(x) = -20 + \frac{25}{2}\left(x^2+(1200)^2\right)^{-1/2}(2x) = 0$$

$x = 1600$

Evaluate C at crit #'s and endpoints:

$C(0) = 40{,}000 + 30{,}000 = 70{,}000$

$\boxed{C(1600) = 8{,}000 + 50{,}000 = 58{,}000}$

$C(2000) \approx 58{,}309$