28 Monday, November 6

Sigma Notation

Definition (Sigma Notation for Finite Sums). The symbol Σ is used as shorthand to denote a sum of numbers following a pattern:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

The a_k are the **terms** of the sum: a_1 is the first term, a_2 is the second term, a_k is the kth **term**, and a_n is the last term. The variable k is the **index of summation**. When computing the sum, k runs through the values from 1 (determined by the $k = 1$ below the Σ) to n (determined by n above the Σ). The 1 is the **lower limit of summation** and the n is the **upper limit of summation.**

Example. Evaluate the following sums.

(1) $\sum_{k=1}^{5} k^2 = 1 + 4 + 9 + 16 + 25$

$= 55$

(2) $\sum_{k=-1}^{3} (k^3 - 1) = ((-1)^3 - 1) + (0^3 - 1) + (1^3 - 1)$

$+ (2^3 - 1) + (3^3 - 1)$

$= (-2) + (-1) + (0) + 7 + 26$

$= 30$

(3) $\sum_{k=-1}^{1} e^k$ $= e^{-1} + 1 + e$

(4) $\sum_{k=2}^{5} \frac{(-1)^k}{k}$ $= \frac{(-1)^2}{2} + \frac{(-1)^3}{3} + \frac{(-1)^4}{4} + \frac{(-1)^5}{5}$

$= \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}$

$= \frac{30 - 20 + 15 - 12}{60} = \frac{13}{60}$

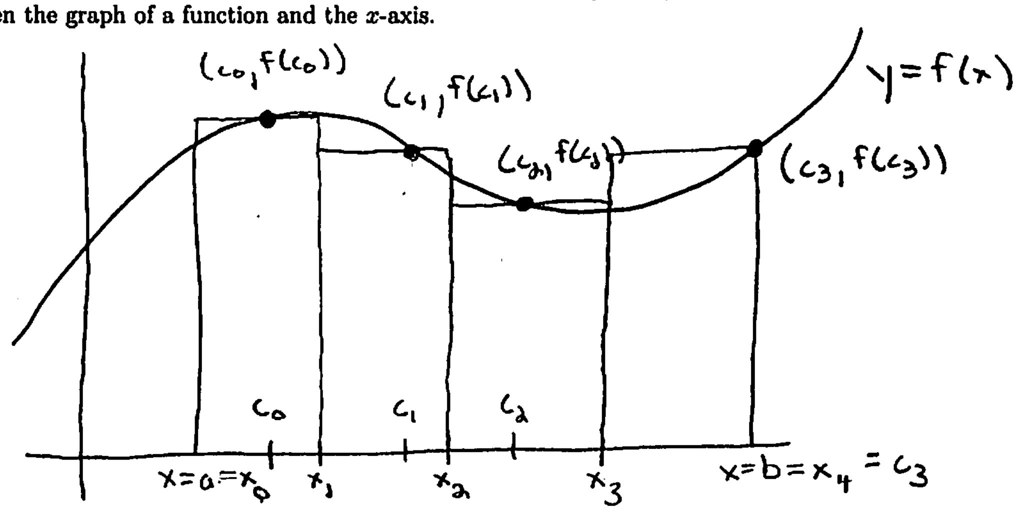
(5) $\sum_{k=0}^{3}\left(3-\left(\frac{2}{3}\right)^k\right)$ $= (3-1) + (3-\frac{2}{3}) + (3-\frac{4}{9}) + (3-\frac{8}{27})$

$= \frac{11 \cdot 27}{27} - \frac{18}{27} - \frac{12}{27} - \frac{8}{27}$

$= \frac{259}{27}$

Riemann Sums

Definition (Riemann Sum). A **Riemann Sum** is a particular summing operation done on a function over a closed interval. The simplest example of a Riemann sum is given by an attempt to approximate the area between the graph of a function and the x-axis.



The area of the region can be approximated by a set of rectangles:

(1) Partition $[a, b]$ into n subintervals of equal size

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\,\Delta x$$

(2) In each subinterval $[x_i, x_{i+1}]$ with $i = 0, 1, \ldots, n-1$, pick c_i. Build a rectangle on the subinterval $[x_i, x_{i+1}]$ up to the point $(c_i, f(c_i))$

(3) The total area of the rectangles is

$$S_n = \sum_{i=0}^{n-1} f(c_i)\,\Delta x \qquad \boxed{\text{Riemann Sum of } f(x)}$$

(4) If the number of rectangles is increased, then the area approximation by the rectangles becomes more accurate

$$A = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(c_i)\,\Delta x$$

Two Particular Riemann Sums

(1) When $c_i = x_i$, we get the left sum: $L_n = \sum_{i=0}^{n-1} f(x_i)\,\Delta x$

(2) When $c_i = x_{i+1}$, we get the right sum: $R_n = \sum_{i=0}^{n-1} f(x_{i+1})\,\Delta x$

reindexing the sum $\quad = \sum_{i=1}^{n} f(x_i)\,\Delta x$

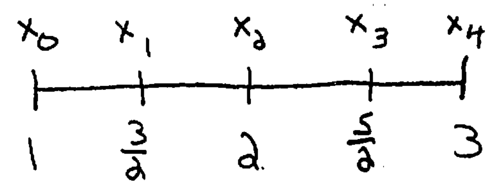
Example. Use the left and right Riemann sums to approximate the area under the following curves.

(1) ~~$y=x$~~ over $[1,3]$ with 4 rectangles

$f(x) = y = x^2$ $\qquad [a,b] \qquad n$

Subinterval Size:

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$$



Left Sum:

$$L_4 = \sum_{i=0}^{4-1} f(x_i)\Delta x$$

$(2x+2y+2z)$

$2(x+y+z)$

$$= \Delta x \sum_{i=0}^{3} f(x_i).$$

$$= \Delta x \left[f(x_0)+f(x_1)+f(x_2)+f(x_3)\right]$$

$$= \frac{1}{2}\left[1+\frac{9}{4}+4+\frac{25}{4}\right]$$

$$= \frac{27}{4}$$

Right Sum:

$$R_4 = \sum_{i=1}^{4} f(x_i)\Delta x$$

$$= \Delta x\left[f(x_1)+f(x_2)+f(x_3)+f(x_4)\right]$$

$$= \frac{1}{2}\left[\frac{9}{4}+4+\frac{25}{4}+9\right]$$

$$= \frac{43}{4}$$

(6) $y = \frac{\sin x}{x}$ over $[1, 11]$ with 50 rectangles

(7) $y = \sqrt{1+x^4}$ over $[0, 50]$ with 200 rectangles

Subinterval Size:

$$\Delta x = \frac{b-a}{n} = \frac{50-0}{200} = \frac{1}{4}$$

$$x_i = a + i\Delta x = 0 + i\left(\frac{1}{4}\right) = \frac{i}{4}$$

Left Sum:

$$L_{200} = \sum_{i=0}^{200-1} f(x_i)\Delta x$$

$$= \sum_{i=0}^{199} \sqrt{1 + x_i^4} \cdot \frac{1}{4}$$

$$= \sum_{i=0}^{199} \sqrt{1 + \frac{i^4}{256}} \cdot \frac{1}{4}$$

Right Sum:

$$R_{200} = \sum_{i=1}^{200} f(x_i)\Delta x = \sum_{i=1}^{200} \sqrt{1 + \frac{i^4}{256}} \cdot \frac{1}{4}$$