3 Monday, August 28

Evaluating Limits

Theorem (Basic Limit Rules).

1. Constant
   \[ \lim_{x \to c} k = k. \]

2. Identity
   \[ \lim_{x \to c} x = c. \]

Now suppose \( \lim_{x \to c} f(x) = L \) and \( \lim_{x \to c} g(x) = M. \) Then

3. Add/Subtract
   \[ \lim_{x \to c} [f(x) \pm g(x)] = L \pm M. \]

4. Constant Multiple (suppose \( k \) is any constant)
   \[ \lim_{x \to c} kf(x) = kL. \]

5. Multiply
   \[ \lim_{x \to c} f(x)g(x) = LM. \]

6. Division
   \[ \lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ whenever } M \neq 0. \]

Applying these four rules give the following results. Suppose \( n \) is an integer.

7. Power
   \[ \lim_{x \to c} x^n = c^n. \]

8. Radical
   \[ \lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}. \]

9. Polynomials
   If \( p(x) \) is a polynomial, then \( \lim_{x \to c} p(x) = p(c). \)

10. Rational Functions
    If \( p(x) \) and \( q(x) \) are polynomials, then
        \[ \lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \]
        if \( q(c) \neq 0. \)
Example. Given \( \lim_{x \to 0} f(x) = 27 \) and \( \lim_{x \to 0} g(x) = 9 \), find the following limits.

1. \( \lim_{x \to 0} [f(x) + g(x)] = 36 \)

2. \( \lim_{x \to 2} 2f(x) = 2(37) = 74 \)

3. \( \lim_{x \to 2} \frac{f(x)}{g(x)} = \frac{\tan 2}{\tan 9} = 3 \)

4. \( \lim_{x \to 2} \frac{f(x) + g(x)}{f(x) - g(x)} = \frac{\lim_{x \to 2} [f(x) + g(x)]}{\lim_{x \to 2} [f(x) - g(x)]} = \frac{37 + 9}{37 - 9} = \frac{36}{18} = 2 \)

Example. Evaluate the following limits.

1. \( \lim_{x \to 2} (-x^2 + 5x - 2) = -4 + 10 - 2 = 4 \)

2. \( \lim_{x \to 2} \frac{x + 3}{x + 6} = \frac{5}{8} \)

3. \( \lim_{h \to 0} \frac{3}{\sqrt{3h + 1} + 1} = \frac{3}{\sqrt{1} + 1} = \frac{3}{2} \)
Theorem 3.1 (Extensions to Power and Radical Rules). Let \( \lim_{x \to c} f(x) = L \). Then

\[(?): \text{Power} \]
\[
\lim_{x \to c} [f(x)]^n = L^n.
\]

\[(?): \text{Radical} \]
\[
\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}.
\]

Example. Evaluate the following limits.

1. \[ \lim_{x \to 1} \sqrt{x^2 + 2x - 4} = \sqrt{\lim_{x \to 1} (x^2 + 2x - 4)} = \sqrt{0 + 0 - 4} = \sqrt{-4} = 2. \]

2. \[ \lim_{x \to 2} (x^3 + 2x^2 + x - 3)^5 = \left( \lim_{x \to 2} \left( x^3 + 2x^2 + x - 3 \right) \right)^5 = (2^3 + 2(2^2) + 2 - 3)^5 = 1. \]

Theorem 3.2 (Limits of Transcendental Functions). Let \( c \) be a number in the domain of the given transcendental function.

1. \[ \lim_{x \to c} \sin x = \sin c. \]
2. \[ \lim_{x \to c} \tan x = \tan c. \]
3. \[ \lim_{x \to c} \sec x = \sec c. \]
4. \[ \lim_{x \to c} \cot x = \cot c. \]
5. \[ \lim_{x \to c} \csc x = \csc c. \]
6. \[ \lim_{x \to c} \ln x = \ln c. \]

Example. Evaluate the following limits.

1. \[ \lim_{x \to 0} \sin x = \sin 0 = 0. \]
2. \[ \lim_{x \to \pi} \sec x = \sec \pi = \frac{1}{\cos \pi} = \frac{1}{-1} = -1. \]
3. \[ \lim_{x \to -\pi/4} \sec x \tan x = \sec \left( -\frac{\pi}{4} \right) \tan \left( -\frac{\pi}{4} \right) = \left( \sqrt{2} \right) \left( -1 \right) = -\sqrt{2}. \]
4. \[ \lim_{x \to 0} \ln (\sec x) = \ln (\sec 0) = \ln 1 = 0. \]
5. \[ \lim_{x \to 3} 2^{x-1} = 2^{3-1} = 2^2 = 4. \]
6. \[ \lim_{x \to \pi} e^{\sin x} = e^{\sin \pi} = e^0 = 1. \]
\[ f(x) = \frac{x^2 - 1}{x - 1} \quad g(x) = x + 1 \]

**Theorem 3.3 (Functions That Agree at All but One Point).** Let \( c \) be a real number, and let \( f(x) = g(x) \) for all \( x \neq c \) in an open interval containing \( c \). If the limit of \( g(x) \) as \( x \to c \) exists, then the limit of \( f(x) \) also exists and

\[
\lim_{x \to c} f(x) = \lim_{x \to c} g(x).
\]

**Definition (Indeterminate Form).** An indeterminate form is a meaningless numerical expression. Examples include:

\[
\begin{array}{cccccc}
\infty & 0 & \infty & \infty & 0^0 & 1^0 & \infty^0
\end{array}
\]

**Example.** Evaluate the following limits.

1. \[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = (x - 1)(x + 1).
\]

   \[\text{indeterminate form} \quad \frac{0}{0} \quad \text{indeterminate form} \quad \lim_{x \to 1} (x + 1) = 2.\]

2. \[
\lim_{u \to 1} \frac{u^4 - 1}{u^3 - 1} = \lim_{u \to 1} \frac{(u^2 + 1)(u^2 - 1)}{(u - 1)(u^2 + u + 1)} = \lim_{u \to 1} \frac{(u^2 + 1)(u + 1)(u - 1)}{(u - 1)(u^2 + u + 1)} = \frac{4}{3}.
\]

3. \[
\lim_{x \to -1} \frac{x^2 + 3x + 2}{x^2 - 1} = \lim_{x \to 1} \frac{(x + 1)(x + 2)}{(x + 1)(x - 1)} = \lim_{x \to 1} \frac{x + 2}{x - 1} = -\frac{1}{3}.
\]

4. \[
\lim_{x \to 2} \frac{x^2 - 7x + 10}{x - 2}
\]
(6) \[ \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \lim_{x \to 9} \frac{(x - 9)}{(x - 9)(\sqrt{x} + 3)} = \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{6}. \]

(7) \[ \lim_{x \to 1} \frac{\sqrt{x} + 2}{\sqrt{x^3} + 2} = \lim_{x \to 1} \frac{\sqrt{x + 2}}{\sqrt{x^3} + 2} = \lim_{x \to 1} \left( \sqrt{x^3} + 2 \right) = 4. \]

(8) \[ \lim_{x \to -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1} \]
Infinite Limits

When evaluating limits of rational functions, one of three cases occur:

(1) The limit returns a number, possibly 0. The value is the limit.

(2) The limit returns the indeterminate form 0/0. Some algebraic rearrangement needs to be done to evaluate the limit, like above.

(3) The limit returns \( \frac{\text{nonzero}}{0} \). In this case, the limit can be \(+\infty\) or \(-\infty\), or the limit may not exist. The result depends on the values of the left-hand and right-hand limits.

Example. Evaluate the following limits.

(1) \( \lim_{x \to -1} \frac{x^2 - 1}{2x + 4} = \frac{0}{0} = 0 \)

(2) \( \lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{x - 2}{(x-2)(x+2)} = \frac{1}{4} \)

(3) \( \lim_{x \to 2} \frac{(x - 2)^2}{x^2 - 4} = \lim_{x \to 2} \frac{(x - 2)^2}{(x-2)(x+2)} = \frac{0}{4} = 0 \)

(4) \( \lim_{x \to 2} \frac{x - 3}{x^2 - 4} = \frac{-1}{0} \)

To determine either \( \pm \infty \), need to determine the sign of \(\frac{x - 3}{x^2 - 4}\) just to the left or right of 2.

Plot on a number line where the factors are zero:

\[
\begin{array}{cccccc}
- & + & - & + & \cdots \\
\longrightarrow & & & & 3 \\
\end{array}
\]

\( \lim_{x \to 2^-} f(x) = + \infty \) \hspace{1cm} \lim_{x \to 2^+} f(x) = - \infty \)
(5) \( \lim_{x \to 0} \frac{x^2 + x - 6}{x^3 - 2x^2} \)

(a) \( x \to 0 \)

(b) \( x \to 2 \)

(6) \( \lim_{x \to 10} \frac{3}{x^{2/5}} \)
(7) \( \lim_{x \to -2} \frac{x^2 - 3x + 2}{x^3 - 4x} \)

(a) \( x \to -2 \)

(b) \( x \to 0 \)

(c) \( x \to 1 \)

(d) \( x \to 2 \)
(8) \( \lim_{x \to \pi/2^+} \tan x \)

(9) \( \lim_{x \to \pi^+} \csc x \)

(10) \( \lim_{x \to \pi/2^-} \sec x \)

(11) \( \lim_{x \to 0^+} \ln x \)
Piecewise Functions

Example. Evaluate the following limits.

(1) Let

$$f(x) = \begin{cases} 
3 - x & x < 2 \\
2 & x = 2 \\
\frac{x}{2} & x > 2 
\end{cases}$$

Find \(\lim_{x \to 3^-} f(x)\) and \(\lim_{x \to -1} f(x)\).

\[
\lim_{x \to -1} (3-x) = 4.
\]

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3-x) = 1
\]
\[
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x}{2} = 1
\]

\(\lim_{x \to 3} f(x) = 1\).

(2) Let

$$f(x) = \begin{cases} 
x + 2 & x < 0 \\
e^x & 0 \leq x \leq 1 \\
2 - x & x > 1 
\end{cases}$$

Find \(\lim_{x \to 0} f(x)\) and \(\lim_{x \to 1} f(x)\).