3 Monday, August 28

Evaluating Limits

Theorem (Basic Limit Rules).

(1) Constant

$\lim_{x \to c} k = k.$

$\lim_{x \to 1} 2 = \lim_{x \to \pi} 2 = \lim_{x \to -100} 2 = 2.$

(2) Identity

$\lim_{x \to c} x = c.$

$\lim_{x \to 1} x = 1$

Now suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then

(3) Add/Subtract

$\lim_{x \to c} [f(x) \pm g(x)] = L \pm M.$

(4) Constant Multiple (suppose k is any constant)

$\lim_{x \to c} kf(x) = kL.$

(5) Multiply

$\lim_{x \to c} f(x)g(x) = LM.$

(6) Division

$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ whenever $M \neq 0$.

Applying these four rules give the following results. Suppose n is an integer.

(7) Power

$\lim_{x \to c} x^n = c^n.$

(8) Radical

$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}.$

(9) Polynomials

If $p(x)$ is a polynomial, then $\lim_{x \to c} p(x) = p(c)$.

(10) Rational Functions

If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

if $q(c) \neq 0$.

Example. Given $\lim_{x\to 2} f(x) = 27$ and $\lim_{x\to 2} g(x) = 9$, find the following limits.

(1) $\lim_{x\to 2}[f(x)+g(x)]$ $= 36$

(2) $\lim_{x\to 2} 2f(x)$ $= 2(27) = 54.$

(3) $\lim_{x\to 2} \frac{f(x)}{g(x)}$ $= \frac{27}{9} = 3$

(4) $\lim_{x\to 2} \frac{f(x)+g(x)}{f(x)-g(x)}$ $= \frac{\lim_{x\to 2}[f(x)+g(x)]}{\lim_{x\to 2}[f(x)-g(x)]} = \frac{27+9}{27-9} = \frac{36}{18} = 2.$

Example. Evaluate the following limits.

(1) $\lim_{x\to 2}(-x^2+5x-2)$ $= -4 + 10 - 2 = 4.$

(2) $\lim_{x\to 2} \frac{x+3}{x+6}$ $= \frac{5}{8}.$

(3) $\lim_{h\to 0} \frac{3}{\sqrt{3h+1}+1}$ $= \frac{3}{\sqrt{1}+1} = \frac{3}{2}.$

Theorem 3.1 (Extensions to Power and Radical Rules). *Let* $\lim_{x \to c} f(x) = L$. *Then*

(7') Power

$$\lim_{x \to c} [f(x)]^n = L^n.$$

(8') Radical

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}.$$

Example. Evaluate the following limits.

(1) $\lim_{x \to 2} \sqrt{x^2 + 2x - 4} = \sqrt{\lim_{x \to 2} (x^2 + 2x - 4)} = \sqrt{4+4-4} = 2.$

(2) $\lim_{x \to 1} (x^3 + 2x^2 + x - 3)^5 = \left(\lim_{x \to 1} [x^3 + 2x^2 + x - 3] \right)^5 = (1+2+1-3)^5$
$= 1.$

Theorem 3.2 (Limits of Transcendental Functions). *Let c be a number in the domain of the given transcendental function.*

(1) $\lim_{x \to c} \sin x = \sin c.$

(2) $\lim_{x \to c} \cos x = \cos c.$

(3) $\lim_{x \to c} \tan x = \tan c.$

(4) $\lim_{x \to c} \cot x = \cot c.$

(5) $\lim_{x \to c} \sec x = \sec c.$

(6) $\lim_{x \to c} \csc x = \csc c.$

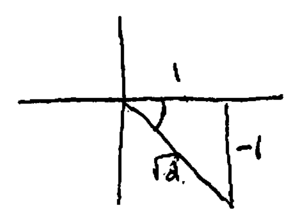
(7) $\lim_{x \to c} a^x = a^c$, $a > 0$.

(8) $\lim_{x \to c} \ln x = \ln c.$

Example. Evaluate the following limits.

(1) $\lim_{x \to 0} \sin x = \sin 0 = 0.$

(2) $\lim_{x \to \pi} \sec x = \sec \pi = \frac{1}{\cos \pi} = \frac{1}{-1} = -1$



(3) $\lim_{x \to -\pi/4} \sec x \tan x = \sec\left(-\frac{\pi}{4}\right) \tan\left(-\frac{\pi}{4}\right) = (\sqrt{2})(-1) = -\sqrt{2}.$

(4) $\lim_{x \to 0} \ln(\sec x) = \ln(\sec 0) = \ln 1 = 0$

(5) $\lim_{x \to 3} 2^{x-1} = 2^{3-1} = 2^2 = 4.$

(6) $\lim_{x \to \pi} e^{\sin x} = e^{\sin \pi} = e^0 = 1$

$f(x) = \dfrac{x^2-1}{x-1}$ $\longrightarrow$ $g(x) = x+1$

Theorem 3.3 (Functions That Agree at All but One Point). *Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as $x \to c$ exists, then the limit of $f(x)$ also exists and*

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$

Definition (Indeterminate Form). An **indeterminate form** is a meaningless numerical expression. Examples include:

$$\boxed{\frac{0}{0}} \quad \frac{\infty}{\infty} \quad 0\cdot\infty \quad \infty-\infty \quad 0^0 \quad 1^\infty \quad \infty^0$$

Example. Evaluate the following limits.

(1) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$

At $x=1$,

$\dfrac{x^2-1}{x-1} = \dfrac{0}{0}$ indeterminate form $\displaystyle = \lim_{x\to 1}(x+1) = 2.$

(2) $\displaystyle\lim_{u\to 1} \frac{u^4-1}{u^3-1} = \lim_{u\to 1} \frac{(u^2+1)(u^2-1)}{(u-1)(u^2+u+1)} = \lim_{u\to 1} \frac{(u^2+1)(u+1)\cancel{(u-1)}}{\cancel{(u-1)}(u^2+u+1)}$

$\displaystyle = \lim_{u\to 1} \frac{(u^2+1)(u+1)}{u^2+u+1} = \frac{4}{3}$

(3) $\displaystyle\lim_{x\to -1} \frac{x^2+3x+2}{x^2-x-2} = \lim_{x\to -1} \frac{\cancel{(x+1)}(x+2)}{\cancel{(x+1)}(x-2)} = \lim_{x\to -1} \frac{x+2}{x-2} = -\frac{1}{3}.$

(4) $\displaystyle\lim_{x\to 2} \frac{x^2-7x+10}{x-2}$

(5) $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$ $\cdot$ $\frac{\sqrt{x}+3}{\sqrt{x}+3}$ $= \lim_{x\to 9.} \frac{\cancel{(x-9)}}{\cancel{(x-9)}(\sqrt{x}+3)}$

$= \lim_{x\to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{6}.$

(6) $\lim_{x\to 1} \frac{\sqrt[3]{x}-1}{x-1}$

(7) $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$ $\cdot$ $\frac{\sqrt{x+3}+2.}{\sqrt{x+3}+2}$ $= \lim_{x\to 1} \frac{\cancel{(x-1)}\left[\sqrt{x+3}+2\right]}{\underbrace{\cancel{(x+3)-4}}_{\cancel{x-1}}}$

$= \lim_{x\to 1}\left(\sqrt{x+3}+2\right) = 4.$

(8) $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$

Infinite Limits

When evaluating limits of rational functions, one of three cases occur:

(1) The limit returns a number, possibly 0. The value is the limit.

(2) The limit returns the indeterminate form 0/0. Some algebraic rearrangement needs to be done to evaluate the limit, like above.

(3) The limit returns $\dfrac{\text{nonzero}}{0}$. In this case, the limit can be $+\infty$ or $-\infty$, or the limit may not exist. The result depends on the values of the left-hand and right-hand limits.

Example. Evaluate the following limits.

(1) $\lim_{x\to 1} \dfrac{x^2-1}{2x+4} = \dfrac{0}{6} = 0$

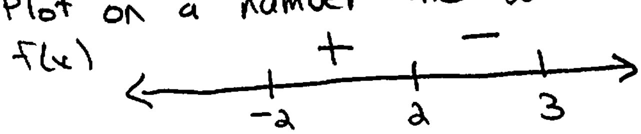
$\dfrac{0}{0}$. (2) $\lim_{x\to 2} \dfrac{x-2}{x^2-4} = \lim_{x\to 2} \dfrac{\cancel{x-2}}{\cancel{(x-2)}(x+2)} = \dfrac{1}{4}$.

$\dfrac{0}{0}$ (3) $\lim_{x\to 2} \dfrac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \dfrac{(x-2)^{\cancel{2}}}{\cancel{(x-2)}(x+2)} = \dfrac{0}{4} = 0$.

(4) $\lim_{x\to 2} \dfrac{x-3}{x^2-4} = \dfrac{-1}{0}$ $\qquad \dfrac{x-3}{(x-2)(x+2)}$.

To determine either $\pm\infty$, need to determine the sign of $\dfrac{x-3}{x^2-4}$ just to the left or right of 2.

Plot on a number line where the factors are zero



$\lim_{x\to 2^-} f(x) = +\infty$ $\qquad \lim_{x\to 2^+} f(x) = -\infty$

$\lim_{x\to 2} f(x) = DNE$

(5) $\lim \dfrac{x^2+x-6}{x^3-2x^2}$

(a) $x \to 0$

(b) $x \to 2$

(6) $\lim\limits_{x\to 0} \dfrac{3}{x^{2/5}}$

(7) $\lim \dfrac{x^2 - 3x + 2}{x^3 - 4x}$

(a) $x \to -2$

(b) $x \to 0$

(c) $x \to 1$

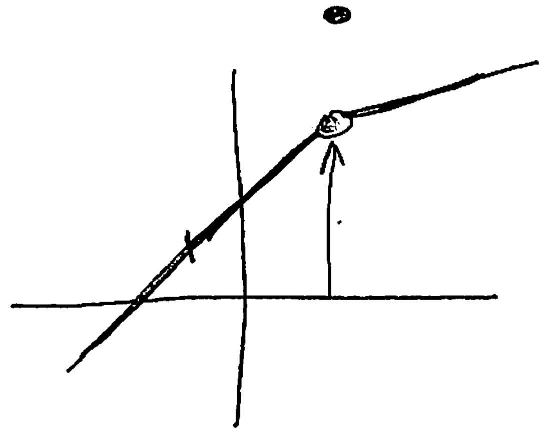
(d) $x \to 2$

(8) $\lim_{x \to \pi/2^-} \tan x$

(9) $\lim_{x \to \pi^+} \csc x$

(10) $\lim_{x \to \pi/2} \sec x$

(11) $\lim_{x \to 0^+} \ln x$



Piecewise Functions

Example. Evaluate the following limits.

(1) Let

$$f(x) = \begin{cases} 3 - x & x < 2. \\ 2 & x = 2 \\ \frac{x}{2} & x > 2 \end{cases}$$

Find $\lim_{x \to 2} f(x)$ and $\lim_{x \to -1} f(x)$.

$$\lim_{x \to -1} (3 - x) = 4.$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3 - x) = 1$$

$$\lim_{x \to 2} f(x) = 1.$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x}{2} = 1$$

(2) Let

$$f(x) = \begin{cases} x + 2 & x < 0 \\ e^x & 0 \le x \le 1 \\ 2 - x & x > 1 \end{cases}$$

Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$.