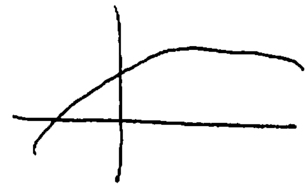
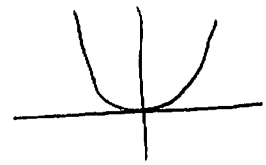


4 Wednesday, August 30



Continuity

Definition (Continuity). A function f is **continuous at** $x = c$ if

(i) $f(c)$ exists.

(ii) $\lim_{x \to c} f(x)$ exists.

(iii) $\lim_{x \to c} f(x) = f(c)$.

Otherwise, f is **discontinuous at** $x = c$. Furthermore, if f is continuous at every x in the open interval (a, b), then f is continuous on (a, b). If f is continuous on (a, b) and

from the right $\qquad \lim_{x \to a^+} f(x) = f(a) \qquad \lim_{x \to b^-} f(x) = f(b), \qquad$ from the left

then f is continuous on $[a, b]$.

Note. Graphically, this definition means that f is continuous on an interval if and only if the graph of f can be drawn with a single, unbreaking stroke.

Example. Examples of continuous functions:

(1) Polynomials are continuous everywhere.

$$f(x) = x + 1 \qquad g(x) = x^2 + 3 \qquad h(x) = x^{100} - x$$

(2) Rational functions are continuous on their domains.

$$f(x) = \frac{x+1}{x-2} \qquad (-\infty, 2) \cup (2, \infty)$$

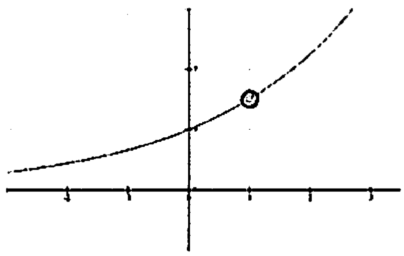
$$g(x) = \frac{x}{x^2 + 3x + 2} \qquad (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$$

(3) If f and g are continuous at $x = c$, then kf (k a real number), $f \pm g$, fg, and $\frac{f}{g}$ ($g(c) \neq 0$) are continuous at $x = c$.

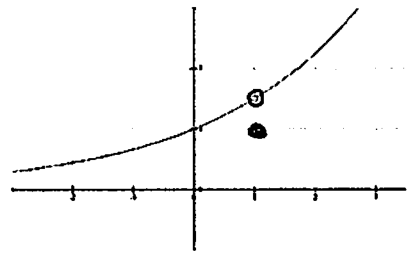
(4) Trigonometric, exponential, and logarithmic functions are all continuous everywhere on their domain.

Example. Examples of discontinuous functions:

(1) Gap/hole discontinuities

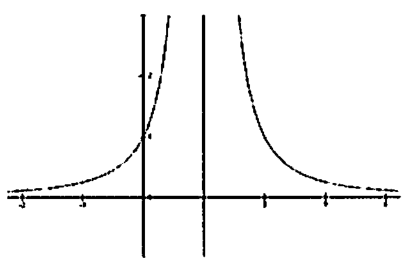


fails (i)

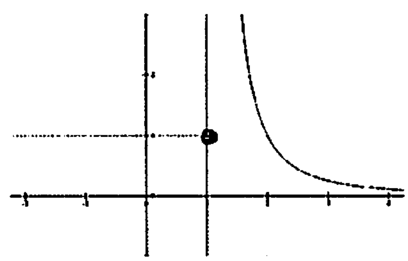


fails (iii).

(2) Infinite discontinuities

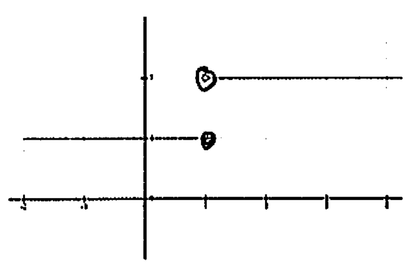


fails (i).



fails (ii)

(3) Jump discontinuities



fails (ii).

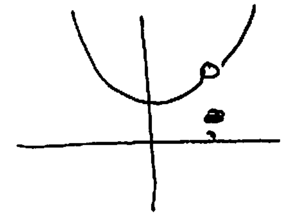
Continuity of Piecewise Functions

To find where a piecewise function is continuous:

(1) Check where each branch is continuous as a function on its own. Compare this with the interval that the branch is defined, taking only the relevant part.

(2) Use the definition of continuity to check the continuity of points where the branches change.

Example. Find the points of discontinuity for each of the following functions. Are the function continuous from the left or right at these points?



(1) $f(x) = \begin{cases} x^2 - 3x + 1 & x \neq 3 \\ 2 & x = 3 \end{cases}$

Each branch is continuous on its own.

Branch Points.

$x = 3$: $f(3) = 2$

$\lim_{x \to 3^-} f(x) = 9 - 9 + 1 = 1.$

$\lim_{x \to 3^+} f(x) = 1$

$\Big\}\ \lim_{x \to 3} f(x) = 1$

$\lim_{x \to 3} f(x) \neq f(3)$

discontinuous at $x=3$

gap.

(2) $f(x) = \begin{cases} 4x + 5 & x \leq -1 \\ x^2 + 1 & x > -1 \end{cases}$

thier

(3) $f(x) = \begin{cases} x^2 - 1 & -1 \le x \le 0 \\ 2x & 0 < x < 1 \\ 1 & x = 1 \\ -2x + 4 & 1 < x < 2 \\ 0 & 2 < x < 3 \end{cases}$

The branches are continuous on their own.

<u>Branch Points</u>

$x = 0$: $f(0) = -1$.

$$\left.\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} (x^2 - 1) = -1 \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} 2x = 0 \end{aligned}\right\} \quad \lim_{x \to 0} f(x) = \text{DNE}$$

discontinuous at $x = 0$:
jump

$x = 1$: $f(1) = 1$.

$$\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} 2x = 2 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (-2x + 4) = 2 \end{aligned}\right\} \quad \lim_{x \to 1} f(x) = 2.$$

$f(1) \neq \lim_{x \to 1} f(x)$.

discontinuous at $x = 1$
gap

$x = 2$: $f(2) =$ undefined

discontinuous at $x = 2$.
gap