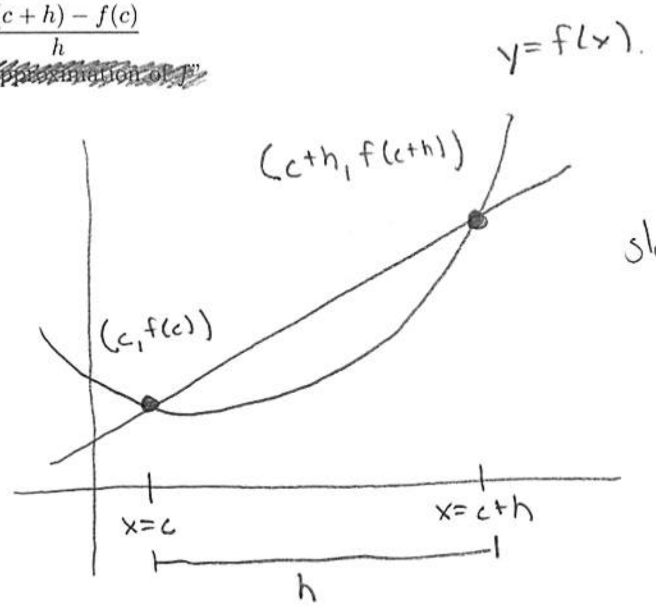
5 Friday, September 1

Derivatives

Definition (Rates of Change). The **average rate of change** of a function f is defined to be the slope of **secant lines**, or lines that meet a graph in at least two points.

$$\begin{aligned} Rate_{avg} &= \text{slope of secant line} \\ &= \frac{\text{change in } f}{\text{change in } x} \\ &= \frac{f(c+h)-f(c)}{h} \end{aligned}$$

~~approximation of f'~~



$$\text{slope} = \frac{f(c+h)-f(c)}{(c+h)-c} = \frac{f(c+h)-f(c)}{h}$$

$\uparrow$

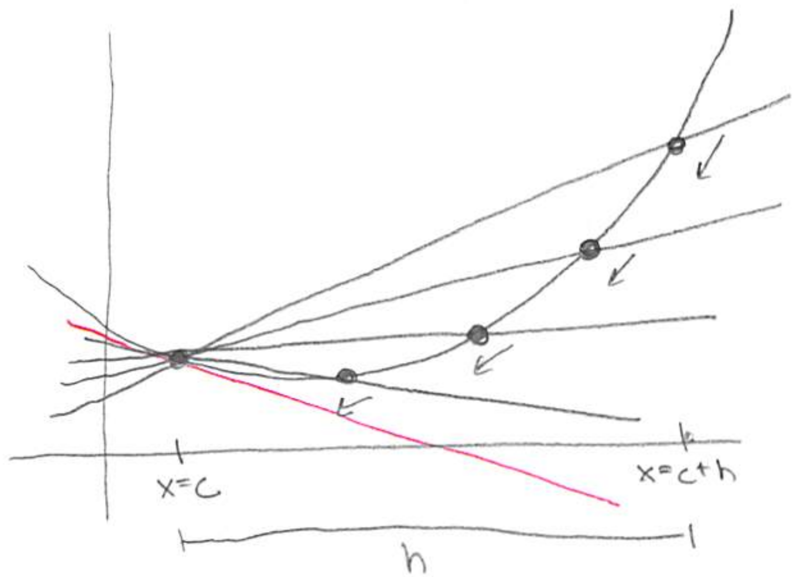
difference quotient

The expression

$$\frac{f(c+h)-f(c)}{h}$$

is called a **difference quotient**.

When it comes to measuring how variables change with respect to one another, the average rate of change does have a flaw, namely that it is an average. Knowing the rate of change at a moment requires looking at **tangent lines**, or lines that meet a graph at a point, but do not cut the graph. Tangent lines are the result of a limiting process where the two points of a secant line come together.



Thus, to find the slope of a tangent line, take a limit as $h \to 0$:

$$\begin{aligned} f'(c) &= \text{slope of tangent line at } x = c \\ &= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \end{aligned}$$

The limiting value is called the **derivative** of the function f at the point $x = c$. The slope of the tangent line is often called the **instantaneous rate of change**.

Note (Notation of derivatives). There are many different ways to denote the derivative:

$$f'(x) \quad \frac{dy}{dx} \quad \frac{df}{dx} \quad y' \quad D_x[y] \quad \left.\frac{dy}{dx}\right|_{x=c}.$$

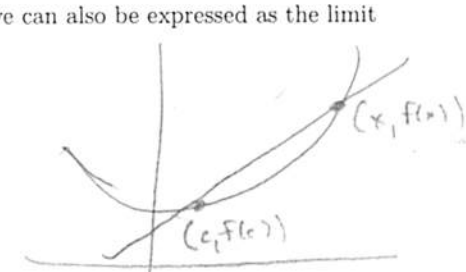
$y = \frac{\Delta y}{\Delta x}$

If $y = x^2 + x$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^2 + x\right) = 2x + 1 \qquad \left.\frac{dy}{dx}\right|_{x=3} = 7.$$

Definition (Alternate definition of the derivative). The derivative can also be expressed as the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$



$\frac{f(x) - f(c)}{x - c}$

Example (Finding derivatives of functions). Using the definition of the derivative, find $f'(x)$.

(1) $f(x) = -3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{-3 - (-3)}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0$$

	0	0.001	0.01	0.1
	-	0	0	0

$$= 0$$

(2) $f(x) = 2x^2 - x + 3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[2(x+h)^2 - (x+h) + 3] - [2x^2 - x + 3]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{-x} - h \cancel{+3} \cancel{-2x^2} \cancel{+x} \cancel{-3}}{h}$$

$$= \lim_{h \to 0} (4x + 2h - 1)$$

$$\boxed{f'(x) = 4x - 1}$$

(3) $f(x) = \frac{3}{6-x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3}{6-(x+h)} - \frac{3}{6-x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3(6-x)}{[6-(x+h)](6-x)} - \frac{3(6-(x+h))}{[6-(x+h)](6-x)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{\cancel{18} - \cancel{3x} - \cancel{18} + \cancel{3x} + 3h}{[6-x-h](6-x)}}{\frac{h}{1}}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{(6-x-h)(6-x)} \cdot \frac{1}{\cancel{h}} = \frac{3}{(6-x)^2}$$

(4) $f(x) = \sqrt{2x+1}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}}$$

$$= \lim_{h\to 0} \frac{[2(x+h)+1] - (2x+1)}{h\left[\sqrt{2(x+h)+1} + \sqrt{2x+1}\right]}$$

$$= \lim_{h\to 0} \frac{2\cancel{h}}{\cancel{h}\left[\sqrt{2(x+h)+1} + \sqrt{2x+1}\right]}$$

$$= \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

(5) $f(x) = \frac{1}{\sqrt{x}}$

Example (Finding derivatives and tangent lines). What is the equation for the line tangent to the graph of f at the specified point?

(1) $f(x) = x^3 - x$ at $c = -1$.

The tangent line goes through $(-1, f(-1)) = (-1, 0)$.

$$f'(-1) = \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h}.$$

$$= \lim_{h \to 0} \frac{(h-1)^3 - (h-1) - 0}{h}.$$

$$= \lim_{h \to 0} \frac{h^3 - 3h^2 + 3h \cancel{-1} - h \cancel{+1}}{h}$$

$$= \lim_{h \to 0} (h^2 - 3h + 2) = 2.$$

Equation for line through $(-1, 0)$ with $m = 2$.

$$y - y_0 = m(x - x_0).$$

$$y - 0 = 2(x - (-1))$$

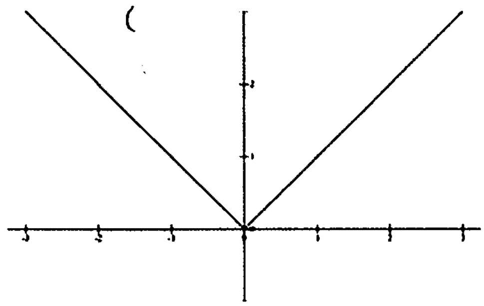
$$\boxed{y = 2x + 2.}$$

(2) $y = \dfrac{x-1}{x+1}$ at $c = 2$.

Note. The derivative can possibly not exist. There are two primary cases for when this happens.

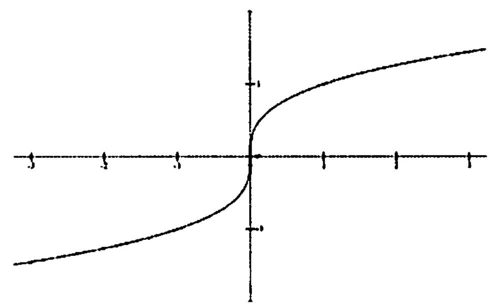
(1) Corners

$f(x) = |x|$



(2) Vertical tangents

$f(x) = \sqrt[3]{x}$



Left and right hand limits do not agree

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

$$(h<0) \quad \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{|h|}{h} = -1$$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{|h|}{h} = 1$$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(c)}{h} = \text{DNE}$$

Derivative is infinite

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^{1/3}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h^{2/3}}$$

$$= +\infty$$