7 Friday, September 8

Review (Rates of Change). The average rate of change of a function $f(x)$ with respect to $x$ over the interval $x_0$ to $x_0 + \Delta x$ is

$$\text{Rate}_{avg} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$ 

The (instantaneous) rate of change of $f$ with respect to $x$ at $x_0$ is the derivative $f'(x_0)$.

Example.

(1) How fast is the area of a circle changing with respect to the radius when the radius is $r = 5$? What is the average rate of change from $r = 5$ to $r = 5.5$?

$$A(r) = \pi r^2$$
$$A'(r) = 2\pi r$$
$$A'(5) = 10\pi$$

$$\text{avg} = \frac{A(5.5) - A(5)}{5.5 - 5} = \frac{30.25\pi - 35\pi}{0.5} = 10.5\pi$$

(2) The number of gallons of water in a tank $t$ minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out the end at 10 min? What is the average rate at which the water flows out during the first 10 min?

$$Q(t) = 200(30 - t)^2$$
$$Q'(t) = 200(30 - 2t)$$
$$Q'(10) = 200(-40) = -8000 \text{ gal/min}$$

$$\text{avg} = \frac{Q(10) - Q(0)}{10 - 0} = \frac{200(30)^2 - 200(30)^2}{10} = \frac{-8000}{10} = -800 \text{ gal/min}$$

(3) Suppose the cost of producing $x$ washing machines is $c(x) = 2000 + 100x - 0.1x^2$. What is the average cost per machine of producing the first 100 machines? What is the rate of change of the cost when 100 machines are produced?

$$\text{avg} = \frac{\text{total cost}}{\# \text{ machines}} = \frac{c(100)}{100} = \frac{2000 + 10000 - 1000}{100} = 110 \text{ }\$/\text{machine}$$

$$c'(x) = 100 - 0.2x$$
$$c'(100) = 80 \text{ }\$/\text{machine}$$
Definition (Displacement and Velocity). Suppose that an object is moving along a line such that its position at time \( t \) is given by \( s = f(t) \). The displacement of the object over the time interval \( t \) to \( t + \Delta t \) is

\[
\Delta s = f(t + \Delta t) - f(t),
\]

and the average velocity of the object is the average rate of change of the position, or

\[
v_{avg} = \frac{\text{displacement}}{\text{time interval}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.
\]

The (instantaneous) velocity is the derivative of the position \( s = f(t) \),

\[
v(t) = \frac{ds}{dt}.
\]

The speed is the absolute value of velocity, \( |v(t)| \).

Example.

(1) A rock is blasted straight up in the air, and its height above the ground is given by \( s = 160t - 16t^2 \) feet.

(a) How high does the rock go?

\[
v(t) = \frac{ds}{dt} = 160 - 32t = 0
\]

\[t = 5\]

\[s(5) = 800 - 400 = 400 \text{ ft}\]
(b) What is the velocity and speed of the rock at \( t = 1 \) and \( t = 6 \) sec? What is the rock's average velocity in this time interval?

\[
\begin{align*}
v(1) &= 128 \text{ ft/s} \\
\text{Speed} &= 128 \text{ ft/s} \\
v(6) &= -32 \text{ ft/s} \\
\text{Speed} &= 32 \text{ ft/s}
\end{align*}
\]

\[
\text{avg} = \frac{s(6) - s(1)}{6 - 1} = \frac{[960 - 576] - [160 - 16]}{5} = \frac{344}{5} = 68.8 \text{ ft/s}
\]

(c) What is the velocity and speed of the rock when it is 256 feet above the ground on the way up? on the way down?

\[
\begin{align*}
\text{Find } t \text{ at this instant:} \\
256 &= 160t - 16t^2 \\
o &= 16t^2 - 160t + 256 \\
o &= t^2 - 10t + 16 \\
o &= (t - 2)(t - 8) \\
t &= 2, 8.
\end{align*}
\]

\[
\begin{align*}
v(2) &= 96 \text{ ft/s} \\
\text{Speed} &= 96 \text{ ft/s} \\
v(8) &= -96 \text{ ft/s} \\
\text{Speed} &= 96 \text{ ft/s}.
\end{align*}
\]

(d) When does the rock hit the ground again?

\[
\begin{align*}
\text{Ground} \equiv s &= 0: \\
o &= 160t - 16t^2 \\
o &= 16t(10 - t) \\
t &= 0, 10.
\end{align*}
\]
2. An astronaut throws a ball up in the air from the surface of the moon. Its height is given by \( s = 24t - 0.8t^2 \) meters in \( t \) seconds.

(a) Find the ball's velocity at time \( t \).

\[
v(t) = 24 - 1.6t
\]

(b) How long does it take the ball to reach its highest point?

Set \( v = 0 \):

\[
0 = 24 - 1.6t
\]

\[
t = 15 \text{ s}
\]

(c) How high does the ball go?

\[
s(15) = 360 - 180 = 180 \text{ m}
\]

(d) How long does it take the ball to reach half its maximum height?

Set \( s = \frac{180}{2} = 90 \):

\[
90 = 24t - 0.8t^2
\]

\[
0 = 0.8t^2 - 24t + 90
\]

\[
0 = t^2 - 30t + 112.5
\]

(e) How long is the ball aloft?

Set \( s = 0 \):

\[
0 = 24t - 0.8t^2
\]

\[
o = t(24 - 0.8t)
\]

\[
t = 15 \text{ s}
\]
(3) A body moving on a coordinate line has position given by $s = 2 - 2\sin t$. What is the body's velocity and speed at time $t = \pi/4$? What is its average velocity from $t = 0$ to $t = \pi/4$.

$v(t) = -2 \cos t$

$v(\pi/4) = -2 (\frac{\sqrt{2}}{2}) = -\sqrt{2}$

speed $= \sqrt{2}$

$v_{avg} = \frac{s(\pi/4) - s(0)}{\pi/4 - 0} = \frac{(2 - \sqrt{2}) - 2}{\pi/4}$

$= -\frac{\sqrt{2}}{\pi} = -\frac{4\sqrt{2}}{\pi}$

(4) Another body has position $s = \sin t + \cos t$. What is the body's velocity and speed at time $t = \pi/3$? What is its average velocity from $t = 0$ to $t = \pi/3$?