Notes

7 Friday, September 8

Review (Rates of Change). The **average rate of change** of a function $f(x)$ with respect to x over the interval x_0 to $x_0 + \Delta x$ is

$$\text{Rate}_{\text{avg}} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

The **(instantaneous) rate of change** of f with respect to x at x_0 is the derivative $f'(x_0)$.

Example.

(1) How fast is the area of a circle changing with respect to the radius when the radius is $r = 5$? What is the average rate of change from $r = 5$ to $r = 5.5$?

$A(r) = \pi r^2$

$A'(r) = 2\pi r$

$A'(5) = 10\pi$

$\text{avg} = \frac{A(5.5) - A(5)}{5.5 - 5}$

$= \frac{30.25\pi - 25\pi}{0.5}$

$= 10.5\pi$

(2) The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out the end at 10 min? What is the average rate at which the water flows out during the first 10 min?

$Q(t) = 200(t^2 - 60t + 900)$

$Q'(t) = 200(2t - 60)$

$Q'(10) = 200(-40)$

$= -8000$ gal/min

$\text{avg} = \frac{Q(10) - Q(0)}{10 - 0}$

$= \frac{200(20)^2 - 200(30)^2}{10}$

$= 20[400 - 900] = -10{,}000$ gal/min

(3) Suppose the cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$. What is the average cost per machine of producing the first 100 machines? What is the rate of change of the cost when 100 machines are produced?

$\text{avg} = \frac{\text{total cost}}{\#\text{ machines}}$

$= \frac{c(100)}{100}$

$= \frac{2000 + 10{,}000 - 1000}{100}$

$= 110$ \$/machine

$c'(x) = 100 - 0.2x$

$c'(100) = 80$ \$/machine

Definition (Displacement and Velocity). Suppose that an object is moving along a line such that its position at time t is given by $s = f(t)$. The **displacement** of the object over the time interval t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t),$$

and the **average velocity** of the object is the average rate of change of the position, or

$$v_{\text{avg}} = \frac{\text{displacement}}{\text{time interval}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

The **(instantaneous) velocity** is the derivative of the position $s = f(t)$,

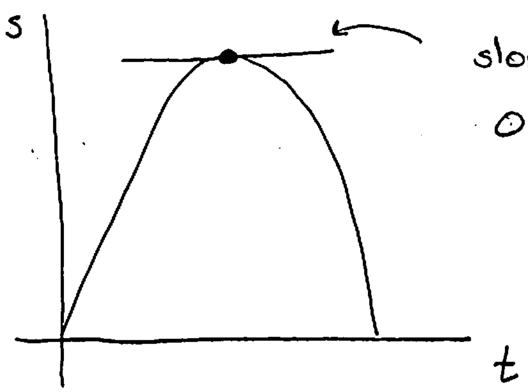
$$v(t) = \frac{ds}{dt}.$$

The **speed** is the absolute value of velocity, $|v(t)|$.

Example.

(1) A rock is blasted straight up in the air, and its height above the ground is given by $s = 160t - 16t^2$ feet.

(a) How high does the rock go?



slope of tangent is 0 at peak.

$$v(t) = \frac{ds}{dt} = 160 - 32t = 0$$

$$t = 5$$

$$s(5) = 800 - 400 = 400 \text{ ft}$$

(b) What is the velocity and speed of the rock at $t = 1$ and $t = \cancel{5}\ 6$ sec? What is the rock's average velocity in this time interval?

$v(t) = 160 - 32t$

$v(1) = 128$ ft/s

speed $= 128$ ft/s

$v(6) = -32$ ft/s

speed $= 32$ ft/s

$$\text{avg} = \frac{s(6) - s(1)}{6 - 1} = \frac{[960 - 576] - [160 - 16]}{5} = \frac{240}{5} = 48 \text{ ft/s}$$

(c) What is the velocity and speed of the rock when it is 256 feet above the ground on the way up? on the way down?

Find t at this instant:

$256 = 160t - 16t^2$

$0 = 16t^2 - 160t + 256$

$0 = t^2 - 10t + 16$

$0 = (t-2)(t-8)$

$t = 2, 8.$

up down

$v(2) = 96$ ft/s

speed $= 96$ ft/s

$v(8) = -96$ ft/s

speed $= 96$ ft/s.

(d) When does the rock hit the ground again?

Ground $\equiv s = 0$:

$0 = 160t - 16t^2$

$0 = 16t(10 - t)$

$\boxed{t = 0, 10}$

(2) An astronaut throws a ball up in the air from the surface of the moon. Its height is given by $s = 24t - 0.8t^2$ meters in t seconds.

(a) Find the ball's velocity at time t.

$v(t) = 24 - 1.6t$

(b) How long does it take the ball to reach its highest point?

Set $v=0$:

$0 = 24 - 1.6t$

$t = 15$ s

(c) How high does the ball go?

$s(15) = 360 - 180 = 180$ m

(d) How long does it take the ball to reach half its maximum height?

Set $s = \frac{180}{2} = 90$:

$90 = 24t - 0.8t^2$

$0 = 0.8t^2 - 24t + 90$

$0 = t^2 - 30t + 112.5$

$t = \frac{30 \pm \sqrt{900 - 450}}{2}$

$= 15 \pm \frac{1}{2}\sqrt{450}$

$= 15 \pm \frac{15}{2}\sqrt{2}$.

(e) How long is the ball aloft?

Set $s=0$:

$0 = 24t - 0.8t^2$

$0 = t(24 - 0.8t)$

$t = 0, 30$

(3) A body moving on a coordinate line has position given by $s = 2 - 2\sin t$. What is the body's velocity and speed at time $t = \pi/4$? What is its average velocity from $t = 0$ to $t = \pi/4$.

$$v(t) = -2\cos t$$

$$v\left(\tfrac{\pi}{4}\right) = -2\left(\tfrac{\sqrt{2}}{2}\right) = -\sqrt{2}$$

speed $= \sqrt{2}$

$$v_{avg} = \frac{s(\frac{\pi}{4}) - s(0)}{\frac{\pi}{4} - 0} = \frac{(\cancel{2} - \sqrt{2}) - \cancel{2}}{\frac{\pi}{4}}$$

$$= \frac{-\sqrt{2}}{\frac{\pi}{4}} = \frac{-4\sqrt{2}}{\pi}$$

(4) Another body has position $s = \sin t + \cos t$. What is the body's velocity and speed at time $t = \pi/3$? What is its average velocity from $t = 0$ to $t = \pi/3$?