8 Monday, September 11

Theorem (Product Rule).

$$\frac{d}{dx}[fg] = \frac{df}{dx}g + \frac{dg}{dx}f$$

$$\frac{d}{dx}(fg) \neq f' \cdot g'$$

Theorem (Quotient Rule).

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) \neq \frac{f'}{g'} \qquad \frac{\text{lo d-high} - \text{high d-lo}}{\text{lo}^2}$$

Theorem (Derivatives of Trigonometric Functions).

$\frac{d}{dx}\sin x = \cos x$

(1) $\frac{d}{dx}[\tan x] = \sec^2 x$

(2) $\frac{d}{dx}[\sec x] = \sec x \tan x$

$\frac{d}{dx}\cos x = -\sin x$

(3) $\frac{d}{dx}[\cot x] = -\csc^2 x$

(4) $\frac{d}{dx}[\csc x] = -\csc x \cot x$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Example. Find the derivative.

(1) $y = \underbrace{(3-x^2)}_{f}\underbrace{(x^3-x+1)}_{g}$

$f' = -2x$

$g' = 3x^2 - 1$

$$y' = \underbrace{(-2x)}_{f'}\underbrace{(x^3-x+1)}_{g} + \underbrace{(3x^2-1)}_{g'}\underbrace{(3-x^2)}_{f}$$

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(2) $y = \underbrace{(x^2+1)}_{f}\underbrace{\left(x+5+\frac{1}{x}\right)}_{g}$

$f' = 2x$

$g' = 1 - \frac{1}{x^2}$

$$y' = \underbrace{2x}_{f'}\underbrace{\left(x+5+\frac{1}{x}\right)}_{g} + \underbrace{\left(1-\frac{1}{x^2}\right)}_{g'}\underbrace{(x^2+1)}_{f}$$

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(3) $y = \underbrace{(\csc x - 6)}_{f}\underbrace{(\sqrt{x}+7)}_{g}$

$$y' = \underbrace{(-\csc x \cot x)}_{f'}\underbrace{(x^{1/2}+7)}_{g} + \underbrace{\frac{1}{2}x^{-1/2}}_{g'}\underbrace{(\csc x - 6)}_{f}.$$

(4) $y = x^2\left(\cot x - \frac{1}{x^2}\right) \quad = x^2 \cot x - 1$

$$y = 2x\left(\cot x - \frac{1}{x^2}\right) + x^2\left(-\csc^2 x + \frac{2}{x^3}\right).$$

(5) $y = \overbrace{(\sec x + \tan x)}^{f}\ \overset{g}{\tan x}$

$$y' = \underbrace{(\sec x \tan x + \sec^2 x)}_{f'}\ \underbrace{\tan x}_{g} + \underbrace{\sec^2 x}_{g'}\ \underbrace{(\sec x + \tan x)}_{f}$$

(6) $y = (\sin x + \cos x)\sec x$

$$y' = (\cos x - \sin x)\sec x + (\sin x + \cos x)\sec x \tan x$$

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$\cot^2 x + 1 = \csc^2 x$

$$= 1 - \cancel{\tan x} + \tan^2 x + \cancel{\tan x}$$

$$= 1 + \tan^2 x$$

$$= \sec^2 x$$

$y = \tan x + 1$

$y' = \sec^2 x$

(7) $y = x^2 \sin x + 2x\cos x - 2\sin x$

$$y' = (x^2 \cos x + 2x\cancel{\sin x}) + (-\cancel{2x\sin x} + 2\cancel{\cos x}) - 2\cancel{\cos x}$$

$$y' = x^2 \cos x$$

(8) $y = \overset{f}{x}\overset{g}{e^x} - e^x$

$$y' = (\underbrace{xe^x}_{fg'} + \underbrace{\cancel{1 \cdot e^x}}_{f'g})\ \cancel{-e^x}$$

$$y' = xe^x$$

(9) $y = \overbrace{(x^2 - 2x + 2)}^{f}\, \overset{g}{e^x}$

$$y' = \underbrace{(x^2 \cancel{-2x} \cancel{+2})}_{f}\underbrace{e^x}_{g'} + \underbrace{(\cancel{2x} \cancel{-2})}_{f'}\underbrace{e^x}_{g}$$

$$y' = x^2 e^x$$

(10) $y = \overset{f}{e^x}\,\overset{g}{\sin x} + \overset{f}{e^x}\,\overset{g}{\cos x}$

$$y' = \left(\cancel{\underset{f'}{e^x}\,\underset{g}{\sin x}} + \underset{f}{e^x}\,\underset{g'}{\cos x}\right) + \left(\underset{f'}{e^x}\,\underset{g}{\cos x} - \cancel{\underset{f}{e^x}\,\underset{g'}{\sin x}}\right)$$

$$y' = 2e^x \cos x$$

$$\frac{d}{dx}(fgh) = f'gh + fg'h + fgh'$$

(11) $y = \overbrace{x \sin x}^{f}\,\overbrace{\cos x}^{g}$

$$y' = \underbrace{(x\cos x + \sin x)}_{f'}\underbrace{\cos x}_{g} + \underbrace{x \sin x}_{f}\underbrace{(-\sin x)}_{g'}$$

$$= x \sin x \underset{\substack{\uparrow \\ \cos x}}{(-\sin x)} + x \underset{\substack{\uparrow \\ \sin x}}{\cos x} \cos x + \underset{\substack{\uparrow \\ x}}{1} \cdot \sin x \cos x$$

(12) $y = x^3 e^x \cos x$

$$y' = 3x^2 e^x \cos x + x^3 e^x \cos x + x^3 e^x (-\sin x).$$

Example. Find the equation for the tangent line at the following points.

(1) $y = (x-1)(x^2+x+1)$ at $c = 0$.

$y' = (x-1)(2x+1) + 1\cdot(x^2+x+1)$.

$y'(0) = (-1)(1) + 1\cdot(1) = 0$.

$y(0) = -1$

Eqn for tangent:

$y - (-1) = 0\cdot(x-0)$

$y = -1$

(2) $y = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x} + 1\right)$ at $c = 1$.

$y' = \left(1 - \frac{1}{x^2}\right)\left(x - \frac{1}{x} + 1\right) + \left(x + \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right)$.

$y'(1) = \cancel{(1-1)(1-1+1)} + (1+1)(1+1)$

$= 4$

$y(1) = (1+1)(1-1+1) = 2$.

$y - 2 = 4(x-1)$

$y = 4x - 2$.

(3) $y = (9x^2 - 6x + 2)e^x$ at $c = 0$.

$y' = (9x^2 - 6x + 2)e^x + (18x - 6)e^x$

$y'(0) = 2e^0 - 6e^0 = -4$

$y(0) = 2$

$y - 2 = -4x$

$y = -4x + 2$.

(4) $y = (1 + \sec x)\sin x$ at $c = \pi/4$.

(5) $y = 2e^x \tan x$ at $c = 0$.