1 Wednesday, August 23

Example. Consider the function

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

Examine the behavior of this function near x = 1.

Example. Consider the piecewise-defined function

$$f(x) = \begin{cases} x+1 & x \neq 1 \\ 0 & x = 1 \end{cases}$$

Examine the behavior of this function near x = 1.

Definition (Informal Definition of Limit). Suppose f(x) is defined in an open interval about x = c, except possibly at x = c. If the values of f(x) become arbitrarily close to L as the values of x approach c from both side, s then the **limit** of f(x) as x approaches c is L, or

$$\lim_{x \to c} f(x) = L.$$

Note. In the definition of the limit, f need not be defined at x = c. In fact, if it is defined at x = c, then *ignore that*. The actual value of f(c) has no bearing on the existence or value of $\lim_{x\to c} f(x)$; limits are an evaluation of the expected value of a function based on the values of nearby points. These limits are sometimes called **deleted limits** for this reason.

To find the limit of f(x) as $x \to c$ numerically, simply evaluate the function at several points that grow closer to c. Then look to see if the values are growing closer to a specific number from either side.

Example. Estimate the following limits numerically.

(1) $\lim_{x \to 2} (3x - 5)$

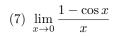
(2) $\lim_{x \to 4} (2x^2 - x + 1)$

(3) $\lim_{x \to -1} (3x^3 - 1)$

(4)
$$\lim_{x \to -1} \frac{1}{(x+1)^2}$$

(5)
$$\lim_{x \to 1} \frac{1}{(x+1)^2}$$

(6)
$$\lim_{x \to 2} \frac{x^2 - 4}{x^2 - 7x + 10}$$



Example. Consider the piecewise-defined function

$$f(x) = \begin{cases} x+2 & x < 0\\ x^2 & x \ge 0 \end{cases}$$

What is the limit as $x \to 0$?

One-Sided Limits

Definition (One-sided Limit). If $f(x) \to L$ as $x \to c$ from x values that are to the left of c (x < c), then

$$\lim_{x \to c^-} f(x) = L.$$

Similarly, if $f(x) \to M$ as $x \to c$ from x values that are to the right of c(x > c), then

$$\lim_{x \to c^+} f(x) = M.$$

Example. The step function

$$H(x) = \begin{cases} 0 & x < 0\\ 1 & x \ge 0 \end{cases}$$

has a left hand limit of 0 and a right hand limit of 1 as $x \to 0$. However, the two-sided limit of this function does not exist since the two sides do not match.

 ${\bf Theorem} \ ({\rm Two-sided} \ {\rm Limit} \ {\rm Existence}).$

$$\lim_{x \to c} f(x) = L \quad \text{if and only if} \quad \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L.$$

Example. Consider the function

$$f(x) = -3x^2 + 1.$$

What are the one-sided limits of f(x) as $x \to 0$? What is the two-sided limit of f(x) as $x \to 0$?

Example. Consider the function

$$f(x) = \frac{6}{3 + e^{1/x}}.$$

What are the one-sided limits of f(x) as $x \to 0$? What is the two-sided limit of f(x) as $x \to 0$?

Example. Consider the function

$$f(x) = \frac{2}{x^2 - 3x + 2}.$$

What are the one-sided limits of f(x) as $x \to 2$? What is the two-sided limit of f(x) as $x \to 2$?

Example. Consider the piecewise-defined function

$$f(x) = \begin{cases} x^2 & 0 \le x < 2\\ x - 1 & x \ge 2 \end{cases}$$

What are the one-sided limits of f(x) as $x \to 2$? What is the two-sided limit of f(x) as $x \to 2$?

Example. Consider the piecewise-defined function

$$f(x) = \begin{cases} x^3 - x & x \le 1\\ (x - 1)^2 & x > 1 \end{cases}$$

What are the one-sided limits of f(x) as $x \to 1$? What is the two-sided limit of f(x) as $x \to 1$?

Example. Consider the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0\\ 0 & x = 0 \end{cases}$$

What are the one-sided limits of f(x) as $x \to 0$? What is the two-sided limit of f(x) as $x \to 0$?