10 Friday, September 15

Chain Rule

Theorem 10.1 (Chain Rule)**.** *Let $y = f(u)$ be differentiable with respect to u, and let $u = g(x)$ be differentiable with respect to x. Then $y = f(g(x))$ is differentiable with respect to x and*

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) \cdot g'(x) \qquad \textit{or} \qquad \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Note. The "u" in this definition is an "auxiliary" variable; when computing dy/dx, u should *not* appear in the final answer.

Example. Given y as a function of u and u as a function of x, find dy/dx.

(1) $y = u^3 + u - 1$
$u = 2x + 1$

(2) $y = \sqrt{u}$
$u = x^2 + 2x - 6$

(3) $y = \dfrac{1}{u+1}$
$u = x^3 - 2x + 5$

(4) $y = (x^3 + 2x^2 + x - 3)^5$

Example. Suppose

$$f(2) = 8 \qquad f'(2) = \frac{1}{3} \qquad g(2) = 2 \qquad g'(2) = -3.$$

Find the derivative $h'(2)$.

(1) $h(x) = 2f(x)$

(2) $h(x) = f(x) + g(x)$

(3) $h(x) = f(x)g(x)$

(4) $h(x) = f(x)/g(x)$

(5) $h(x) = f(g(x))$

(6) $h(x) = \sqrt{f(x)}$

(7) $h(x) = 1/g(x)^2$

(8) $h(x) = \sqrt{f(x)^2 + g(x)^2}$

Theorem 10.2 (General Power Rule)**.** *If $f(x)$ is differentiable and n is a rational number, then*

$$\frac{d}{dx}\left[(f(x))^n\right] = nf(x)^{n-1}f'(x).$$

Example. Find $f'(x)$.

(1) $f(x) = (3x^2 - x + 1)^4$

(2) $f(x) = \sqrt[5]{x^4 - \frac{2}{x}}$

(3) $f(x) = \dfrac{1}{2x^2 - x + 5}$

(4) $f(x) = \dfrac{1}{(5x - 3)^6}$

(5) $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$

(6) $f(x) = (10x - 7)^6(x^2 + 1)^4$

(7) $f(x) = \dfrac{(x-4)^3}{(2x+1)^7}$

(8) $f(x) = \dfrac{\frac{1}{x} + x^2}{(2x+5)^3}$