13 Wednesday, September 27

Implicit Differentiation

The derivative gives the slope of the tangent line on a graph given by a function. But not all graphs are represented by a function (i.e. graph does not pass vertical line test). However, these graphs still have tangent lines. The goal of this section is to find the slope of the tangent line in these graphs.

Definition. If $y = f(x)$, where $f(x)$ is just a function of x (no y's), then y is defined **explicitly**. Otherwise, y is defined **implicitly**.

Example. These equations define y implicitly.

(1) $x^2 + y^2 = 1$ (2) $y^2 = x^3 - 2x + 2$ (3) $x^3 - xy + y^3 = 1$

Note. Implicit definitions do not necessarily define functions. However, functions can satisfy an implicit relation. For example, $y = f(x) = \sqrt{1 - x^2}$ is a function that satisfies $x^2 + y^2 = 1$.

Example 13.1. Suppose $y = f(x)$ satisfies $x^2y + xy^2 = 6$. Find dy/dx. To do so, treat y like a function $f(x)$ and differentiate both sides, applying chain rule when appropriate.

Definition (Implicit Differentiation)**.**

(1) Differentiate the equation treating y like $y(x)$, a function of x (this means that if y is in a term to be differentiated, chain rule or product/quotient rule will be necessary).

(2) Solve for y'.

Example 13.2. Find dy/dx.

(1) $2xy + y^2 = x + y$

(2) $x^2(x-y)^2 = x^2 - y^2$

(3) $(3xy+7)^2 = 6y$

(4) $x^2 = \dfrac{x-y}{x+y}$

(5) $x^3 - xy + y^3 = 1$

(6) $(x^2 + 3y^2)^5 = 2xy$

(7) $x = \sin y$

(8) $x + \cos y = xy$

(9) $x \sin 2y = y \cos 2x$

(10) $x = \ln y$

(11) $x + \ln(xy) = 0$

(12) $\cos x + \cos y = xy$

Example 13.3. Find the equation for the tangent line at the given point.

(1) $x^3 - y^3 = -19$ at $(2, 3)$

(2) $(2x + y)^3 = x$ at $(1, -1)$

(3) $6x^2 + 3xy + 2y^2 + 17y - 6 = 0$ at $(-1, 0)$

(4) $x = \tan y$ at $(1, \pi/4)$

(5) $2xy + \pi \sin y = 2\pi$ at $(1, \pi/2)$

(6) $x^2 \cos^2 y - \sin y = 0$ at $(0, \pi)$