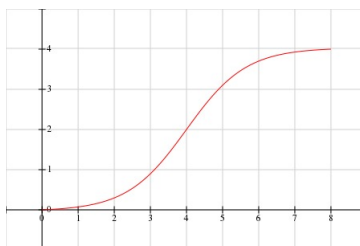


18 Friday, October 13

Concavity

Example. Let $Q(t)$ be the number of units a worker had made after t hours.

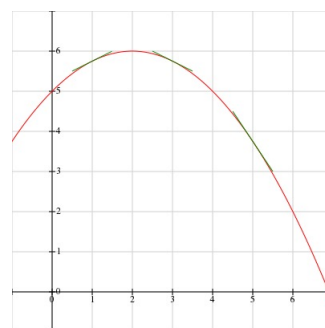
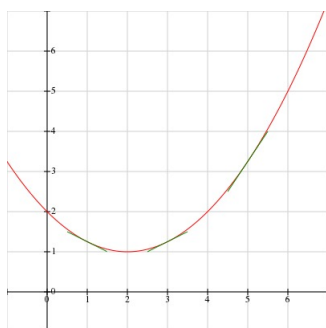


What is the time of his maximal efficiency, i.e. when is he producing units the fastest? His rate of production is the first derivative. To find the maximum of this, use the derivative of the derivative. If there is a maximum at a point, then by the FDT, the first derivative should change from increasing to decreasing.

Definition (Concavity). If f is differentiable on (a, b) , then

(i) f is **concave up** if f' is increasing.

(ii) f is **concave down** if f' is decreasing.



Example. Find the concavity of $f(x) = x^4 - 6x^3 - 5$.

Theorem (Second Derivative Test for Concavity). Suppose f is twice differentiable on (a, b) .

- (i) If $f''(x) > 0$ for all x in (a, b) , then f is concave up.
- (ii) If $f''(x) < 0$ for all x in (a, b) , then f is concave down.

Definition (Inflection Points). A point on the graph of f where f is continuous and the concavity changes.

Finding Intervals of Concavity and Inflection Points:

- (i) Find f'' and the critical numbers of f' . Mark them on a number line.
- (ii) Pick test points from each interval and determine the sign of f'' at those points.
- (iii) Apply the SDT to determine concavity. In addition, any critical number of f' where the concavity changes and f is continuous is an inflection point.

Example. Find inflection points of $f(x) = x^4 - 6x^3 - 5$.

Theorem (Second Derivative Test for Relative Extrema). Suppose $f'(c) = 0$ and $f''(x)$ exists on (a, b) containing c . Then

- (i) If $f''(c) < 0$, then f has a relative maximum at $x = c$.
- (ii) If $f''(c) > 0$, then f has a relative minimum at $x = c$.
- (iii) If $f''(c) = 0$, then the test is inconclusive and FDT must be used.

Note. In the second derivative test for relative extrema, an inconclusive result *does not* imply no extrema occur at that point. See the above example.1

Example 18.1. Find the intervals of increasing/decreasing, concavity, inflection points, and relative extrema for each function.

(1) $f(x) = x^5 - 5x^4$

(2) $f(x) = x^{1/5}$

(3) $f(x) = (x + 1)(x - 1)^2$

$$(4) \quad f(x) = \frac{x^2 - 3}{x - 2}$$

(5) $f(x) = x\sqrt{8 - x^2}$

(6) $f(x) = \frac{x+1}{(x-1)^2}$

(7) $f(x) = \frac{12}{x^2 + 12}$

(8) $f(x) = x + \frac{9}{x}$

(9) $f(x) = (2 - x^2)^{3/2}$

(10) $f(x) = \frac{x^3}{3x^2 + 1}$

(11) $f(x) = x^2 \ln x$

(12) $f(x) = \cos x - 9x, \quad (0, 4\pi)$

(13) $f(x) = \ln \sqrt{x^2 + 4}$

(14) $f(x) = xe^{-7x}$