20 Friday, October 20

Limits at Infinity

Definition (Limits at Infinity)**.** If the values of the function $f(x)$ approach the number L as x grows without bound, then

$$\lim_{x \to \infty} f(x) = L$$

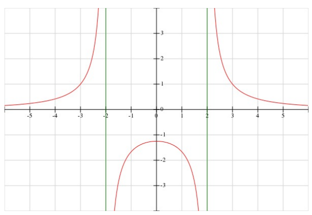
and similarly

$$\lim_{x \to -\infty} f(x) = M.$$

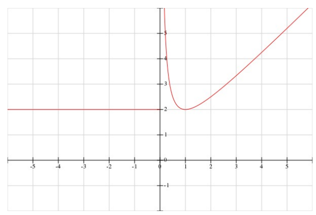
These limiting values, if the exist, create horizontal asymptotes.

Example. Find the horizontal asymptotes of the following graphs.

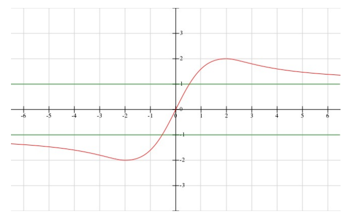
(1)



(2)



(3)



Example. Use a calculator to complete the table and estimate the limit as x approaches infinity.

$$f(x) = \frac{10x + 7}{5x - 3}$$

x	1	10	10^2	10^3	10^4	10^5	10^6
$f(x)$							

Theorem (Reciprocal Limit Rule)**.** If A is a real number, $k > 0$, and x^k is defined for all x, then

$$\lim_{x \to \pm\infty} \frac{A}{x^k} = 0.$$

To find limits at infinity:

(1) Divide each term in the numerator and denominator by the highest power of x in the numerator and denominator.

(2) Apply limit rules.

Example. Find each limit.

(1) $\lim_{x \to \infty} \left(5 + \frac{8}{x}\right)$

(2) $\lim_{x \to \infty} \frac{5 - 7x}{5x^3 - 9}$

(3) $\lim_{x \to \infty} \frac{5 - 7x}{5x - 9}$

(4) $\lim_{x \to \infty} \frac{5 - 7x^2}{5x - 9}$

(5) $\lim_{x\to\infty} \frac{x^2 - 2x + 5}{2x^2 + 5x + 1}$

(6) $\lim_{x\to-\infty} \frac{2x + 1}{3x^2 + 2x - 7}$

(7) $\lim_{x\to-\infty} \frac{9x^3 + 7}{18x^3 - 6x^2 + 4}$

Definition. A rational function $f(x) = p(x)/q(x)$ has a **dominant term** $g(x)$ as $x \to \infty$ (or $x \to -\infty$) if it can be written such that

$$f(x) = g(x) + \frac{r(x)}{q(x)}, \qquad \text{where} \qquad \lim_{x\to\infty} \frac{r(x)}{q(x)} = 0.$$

Dominant terms are functions such that when x is very large, then $f(x) \approx g(x)$ so that $g(x)$ *dominates* the $r(x)/q(x)$ term. In particular, when p and q are polynomials, f has a dominant term when $\deg p \geq \deg q$, and when $\deg p = \deg q + 1$, then f has $g(x)$ for a **slant asymptote**.

Theorem (Rational Function Theorem)**.** Let

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0} \quad \text{with } \deg p = n, \deg q = m$$

be a rational function. Then:

(1) $f(x)$ has a vertical asymptote $x = c$ if $q(c) = 0$ but $p(c) \neq 0$. If both $p(c) = q(c) = 0$, then factor $f(x)$ and perform any cancellations. Check $p(c)$ and $q(c)$ afterwards

(2) $f(x)$ has the following possibilities for a horizontal asymptote:

  (a) if $n > m$, then f has no horizontal asymptotes, and if $n = m + 1$, then f has a slant asymptote.
  (b) if $n < m$, then $y = 0$ is a horizontal asymptote.
  (c) if $n = m$, then $y = \dfrac{a_n}{b_n}$ is a horizontal asymptote.

Example. Find the horizontal and vertical asymptotes of the following functions.

(1) $f(x) = \dfrac{1}{x-1}$

(2) $f(x) = \dfrac{x+3}{x-2}$

(3) $f(x) = \dfrac{x^2+x-6}{x^2-16}$

(4) $f(x) = \dfrac{2x^2+x-1}{x^2-1}$

(5) $f(x) = \dfrac{4x}{x^2 + 4}$

(6) $f(x) = \dfrac{2x^2}{x + 1}$

(7) $f(x) = \dfrac{x^2 - 4}{x - 1}$

(8) $f(x) = \dfrac{x^4 + 1}{x^2}$

(9) $f(x) = \dfrac{x^3 + x - 2}{x - x^2}$

(10) $f(x) = \dfrac{x^3 + 3x + 1}{x^2 + x - 2}$

Example. Find the following limits.

(1) $\displaystyle \lim_{x\to\infty} \frac{5-2x^{3/2}}{3x^2-4}$

(2) $\displaystyle \lim_{x\to-\infty} \frac{x}{\sqrt{x^2-x}}$

(3) $\displaystyle \lim_{x\to-\infty} \frac{\sqrt{x^4-1}}{x^3-1}$

(4) $\lim_{x \to -\infty} \frac{2x}{(x^6 - 1)^{1/3}}$

(5) $\lim_{x \to \infty} \cos x$

(6) $\lim_{x \to \infty} \sin \frac{1}{x}$

(7) $\lim_{x\to\infty} \frac{1}{e^x + 1}$

(8) $\lim_{x\to-\infty} \frac{1}{e^x + 1}$

(9) $\lim_{x \to \pm\infty} \left[x + \sqrt{x^2 + 3} \right]$

(10) $\lim_{x \to \pm\infty} \frac{x}{3x - \sqrt{9x^2 - x}}$

(11) $\lim_{x \to \infty} [\ln(x+1) - \ln x]$