26 Wednesday, November 1

Antiderivatives

Definition. Given a function $f(x)$, a function $F(x)$ such that $F'(x) = f(x)$ is called an **antiderivative** of $f(x)$. For example, the functions

$$x^2 - x + 1 \qquad x^2 - x - 5$$

are antiderivatives of the function $2x - 1$. The process of finding an antiderivative is known as **indefinite integration**. To denote the process of integration, we have the following notation:

$$\int f(x)\,dx = F(x) + C \qquad \text{means} \qquad \frac{d}{dx}F(x) = f(x)$$

The symbol $\int$ is the **integral sign**, the function $f(x)$ is the **integrand**, the variable denoted in the dx is the **variable of integration**, and the C is the **constant of integration**.

An important observation is that, because the derivative of a constant is zero, you can alter one antiderivative of a function $f(x)$ by a constant to get another antiderivative of $f(x)$. In fact, by a corollary of the Mean Value Theorem, the complete set of possible antiderivatives of a function all differ by a constant:

Theorem. If $F(x)$ and $G(x)$ are antiderivatives of a function $f(x)$, then for all x, $F(x) - G(x) = C$ with some constant C.

For this reason, we always write in the constant of integration C when finding the indefinite integral of a function.

For every derivative rule, there is a corresponding antiderivative rule:

Differentiation Rule	Integration Rule
$\frac{d}{dx}C = 0$	$\int 0\,dx = C$
$\frac{d}{dx}kx = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}f(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$
$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
$\frac{d}{dx}x^{n+1} = (n+1)x^n, \quad n \neq -1$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}\ln\lvert x\rvert = \begin{cases} \frac{d}{dx}\ln x & x > 0 \\ \frac{d}{dx}\ln(-x) & x < 0 \end{cases} = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

It is worth noting that, in stark contrast with differentiation, integration is in general a very difficult process. In fact, many functions that are simply expressed just do not have an antiderivative that can be written in a closed form. Such integrals are called **nonelementary integrals**. For example,

$$\int \frac{\sin x}{x}\,dx \qquad \int \sqrt{1-x^4}\,dx \qquad \int \sin\left(x^2\right)dx \qquad \int e^{-x^2}\,dx$$

are all nonelementary integrals. They appear quite frequently in physics, engineering, and statistics. Evaluating such indefinite integrals requires the use of infinite series (a topic discussed in MA16020).

Example. Find the antiderivative of the following functions.

(1) $\displaystyle\int (x^3 - 2x)\, dx$

(2) $\displaystyle\int x^{100}\, dx$

(3) $\displaystyle\int \sqrt{y}\, dy$

(4) $\int \left(x^2 + x^{-2}\right) dx$

(5) $\int e \, dx$

(6) $\int (t+2)(t-3) \, dt$

(7) $\displaystyle\int \frac{x-1}{\sqrt[3]{x}}\,dx$

(8) $\displaystyle\int (2\sin x - e^x)\,dx$

(9) $\displaystyle\int (x^e + 2e^x)\,dx$

(10) $\int x\left(\sqrt[3]{x}+\sqrt[5]{2x}\,dx\right)$

(11) $\int \frac{1+\cos^2 x}{\cos^2 x}\,dx$

(12) $\int \frac{\sqrt{u}-u}{u^2}\,du$

(13) $\int (\theta - 4\csc\theta\cot\theta)\, d\theta$

(14) $\int \frac{\sin 2x}{\sin x}\, dx$

(15) $\int v\left(v^2+2\right)^2 dv$