27 Friday, November 3

Initial Value Problems

Definition. A equation that relates a function to its derivative is called a **differential equation**. A simple example would be the equation

$$\frac{dy}{dx} = 2x \qquad \Longrightarrow \qquad y = x^2 + C$$

Since a function has an infinite set of antiderivatives (all differing by a constant), the solution to a differential equation is called the **general solution**. Real-world problems will often involve trying to find a **particular solution** given some value of the solution function at a particular point. For example,

$$\frac{dy}{dx} = 2x, \quad y(0) = 1 \qquad \Longrightarrow \qquad y = x^2 + 1$$

Such a pairing of a differential equation with an initial value is called an **initial value problem**.

Solving an initial value is simply a matter of finding the general solution and then substituting in the initial value to solve for the constant of integration. Some differential equations involve higher order derivatives. In this case, for every integration done, there must be an initial value point. Therefore, in initial value problems with second order derivatives, for example, there must be two distinct initial values of either the function or its derivatives.

Example. Solve the initial value problem

(1) $\dfrac{dy}{dx} = 10 - x$, $y(0) = -1$

(2) $\dfrac{ds}{dt} = 1 + \cos t,\ s(0) + 4$

(3) $\dfrac{dy}{dx} = 3x^{-2/3},\ y(-1) = -5$

(4) $\dfrac{ds}{dt} = \dfrac{1}{2}\sec t \tan t,\ s(0) = 1$

(5) $\dfrac{dv}{dt} = 8t + \csc^2 t,\ v\left(\dfrac{\pi}{2}\right) = -7$

(6) $\dfrac{d^2y}{dx^2} = 2 - 6x,\ y'(0) = 4, y(0) = 1$

(7) $\dfrac{d^2y}{dx^2} = 0,\ y'(0) = 2, y(0) = 0$

(8) $\dfrac{d^2r}{dt^2} = \dfrac{2}{t^3}$, $r'(1) = 1, r(1) = 1$

(9) $y^{(4)} = -\sin t + \cos t$, $y'''(0) = 7, y''(0) = y'(0) = -1, y(0) = 3$

Example. On the moon, the acceleration of gravity is $1.6\,\mathrm{m/s^2}$. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 seconds later?

Example. You are diving along a highway at a steady 88 ft/s when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 feet?