Review: Riemann Sums

**Definition** (Riemann Sum). A **Riemann Sum** is a particular summing operation done on a function over a closed interval. The simplest example of a Riemann sum is given by an attempt to approximate the area between the graph of a function and the $x$-axis.
Definition (Definite Integral). Let \( f(x) \) be a function defined on a closed interval \([a, b]\). Then the limit of the Riemann sum as the number of partitions increases without bound is the definite integral of \( f(x) \) over \([a, b]\):

\[
\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x = \int_{a}^{b} f(x) \, dx
\]

Note that this limit is independent of the choice of \( c_i \). In the definite integral, \( a \) is the lower limit of integration and \( b \) is the upper limit of integration. \( f(x) \) is the integrand and \( x \) is the variable of integration. The definite integral represents the area under the curve of \( y = f(x) \); as a result, the definite integral is a number and

\[
\int_{a}^{b} f(x) \, dx = \int_{a}^{b} f(t) \, dt = \int_{a}^{b} f(u) \, du.
\]

The variable of integration simply acts as a placeholder; for this reason, it is often referred to as a dummy variable.

Since the definite integral measures the area under the curve of a function, we can use geometric formulas to compute some easier integrals.

Example. Use geometric formulas to compute the definite integrals.

1. \( \int_{-2}^{4} \left( \frac{x}{2} + 3 \right) \, dx \)

2. \( \int_{-1}^{5} 6 \, dx \)
(3) \[ \int_{-3}^{3} \sqrt{9 - x^2} \, dx \]

(4) \[ \int_{-2}^{1} |x| \, dx \]

(5) \[ \int_{-1}^{1} \left( 1 + \sqrt{1 - x^2} \right) \, dx \]
Properties of Definite Integrals

Theorem.

(1) For non-negative functions $f(x)$, the definite integral represents the area between the graph of $y = f(x)$ and the $x$-axis.

(2) For negative functions $f(x)$, the definite integral is the signed area between the $x$-axis and the graph of $x$-axis. If $f(x)$ is a function that is has positive and negative parts, then the definite integral is the positive area minus the negative area.

(3) Zero:

$$\int_a^a f(x) \, dx = 0$$

(4) Order of Integration:

$$\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$$

(5) Constant Multiple: for any $k$ real number,

$$\int_a^b kf(x) \, dx = k \int_a^b f(x) \, dx$$

(6) Sums and Differences:

$$\int_a^b [f(x) \pm g(x)] \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx$$

(7) Additivity:

$$\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$$

regardless of whether $c$ is between $a$ and $b$ or not.
Example. Suppose that
\[ \int_1^2 f(x) \, dx = -4 \quad \int_1^5 f(x) \, dx = 6 \quad \int_1^5 g(x) \, dx = 8. \]
Find the following.

1. \( \int_2^2 g(x) \, dx \)
2. \( \int_1^5 g(t) \, dt \)
3. \( \int_1^2 3f(x) \, dx \)
4. \( \int_2^5 f(u) \, du \)
5. \( \int_1^5 [f(x) - g(x)] \, dx \)
6. \( \int_1^5 [4f(y) - 2g(y)] \, dy \)

Example. Given that
\[ \int_0^b x^2 \, dx = \frac{b^3}{3} \]
find the following.

1. \( \int_3^1 7 \, dx \)
2. \( \int_0^3 (3t + 5) \, dt \)
(3) \( \int_3^0 (2z - 3) \, dz \)

(4) \( \int_0^2 (3x^2 + x - 5) \, dx \)

(5) \( \int_1^2 24u^2 \, du \)