29 Wednesday, November 8

Review: Riemann Sums

Definition (Riemann Sum)**.** A **Riemann Sum** is a particular summing operation done on a function over a closed interval. The simplest example of a Riemann sum is given by an attempt to approximate the area between the graph of a function and the x-axis.

Definition (Definite Integral)**.** Let $f(x)$ be a function defined on a closed interval $[a, b]$. Then the limit of the Riemann sum as the number of partitions increases without bound is the **definite integral of** $f(x)$ **over** $[a, b]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\, dx$$

Note that this limit is *independent* of the choice of c_i. In the definite integral, a is the **lower limit of integration** and b is the **upper limit of integration**. $f(x)$ is the **integrand** and x is the **variable of integration**. The definite integral represents the **area under the curve** of $y = f(x)$; as a result, the definite integral is a number and

$$\int_a^b f(x)\, dx = \int_a^b f(t)\, dt = \int_a^b f(u)\, du.$$

The variable of integration simply acts as a placeholder; for this reason, it is often referred to as a **dummy variable**.

Since the definite integral measures the area under the curve of a function, we can use geometric formulas to compute some easier integrals.

Example. Use geometric formulas to compute the definite integrals.

(1) $\displaystyle\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$

(2) $\displaystyle\int_{-1}^{5} 6\, dx$

(3) $\int_{-3}^{3} \sqrt{9-x^2}\, dx$

(4) $\int_{-2}^{1} |x|\, dx$

(5) $\int_{-1}^{1} \left(1+\sqrt{1-x^2}\right) dx$

Properties of Definite Integrals

Theorem.

(1) For non-negative functions $f(x)$, the definite integral represents the area between the graph of $y = f(x)$ and the x-axis.

(2) For negative functions $f(x)$, the definite integral is the *signed* area between the x-axis and the graph of x-axis. If $f(x)$ is a function that is has positive and negative parts, then the definite integral is the positive area minus the negative area.

(3) Zero:

$$\int_a^a f(x)\,dx = 0$$

(4) Order of Integration:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(5) Constant Multiple: for any k real number,

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

(6) Sums and Differences:

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

(7) Additivity:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

regardless of whether c is between a and b or not.

Example. Suppose that

$$\int_1^2 f(x)\,dx = -4 \qquad \int_1^5 f(x)\,dx = 6 \qquad \int_1^5 g(x)\,dx = 8.$$

Find the following.

(1) $\displaystyle\int_2^2 g(x)\,dx$

(2) $\displaystyle\int_5^1 g(t)\,dt$

(3) $\displaystyle\int_1^2 3f(x)\,dx$

(4) $\displaystyle\int_2^5 f(u)\,du$

(5) $\displaystyle\int_1^5 [f(x) - g(x)]\,dx$

(6) $\displaystyle\int_1^5 [4f(y) - 2g(y)]\,dy$

Example. Given that

$$\int_0^b x^2\,dx = \frac{b^3}{3}$$

find the following.

(1) $\displaystyle\int_3^1 7\,dx$

(2) $\displaystyle\int_0^{-3} (3t + 5)\,dt$

(3) $\int_{3}^{0} (2z - 3)\, dz$

(4) $\int_{0}^{2} \left(3x^2 + x - 5\right) dx$

(5) $\int_{1}^{2} 24u^2\, du$