3 Monday, August 28

Evaluating Limits

Theorem (Basic Limit Rules)**.**

(1) Constant

$\lim_{x \to c} k = k.$

(2) Identity

$\lim_{x \to c} x = c.$

Now suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then

(3) Add/Subtract

$\lim_{x \to c} [f(x) \pm g(x)] = L \pm M.$

(4) Constant Multiple (suppose k is any constant)

$\lim_{x \to c} kf(x) = kL.$

(5) Multiply

$\lim_{x \to c} f(x)g(x) = LM.$

(6) Division

$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$ whenever $M \neq 0$.

Applying these four rules give the following results. Suppose n is an integer.

(7) Power

$\lim_{x \to c} x^n = c^n.$

(8) Radical

$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}.$

(9) Polynomials

If $p(x)$ is a polynomial, then $\lim_{x \to c} p(x) = p(c)$.

(10) Rational Functions

If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$$

if $q(c) \neq 0$.

Example. Given $\lim_{x \to 2} f(x) = 27$ and $\lim_{x \to 2} g(x) = 9$, find the following limits.

(1) $\lim_{x \to 2} [f(x) + g(x)]$

(2) $\lim_{x \to 2} 2f(x)$

(3) $\lim_{x \to 2} \frac{f(x)}{g(x)}$

(4) $\lim_{x \to 2} \frac{f(x) + g(x)}{f(x) - g(x)}$

Example. Evaluate the following limits.

(1) $\lim_{x \to 2} \left(-x^2 + 5x - 2\right)$

(2) $\lim_{x \to 2} \frac{x + 3}{x + 6}$

(3) $\lim_{h \to 0} \frac{3}{\sqrt{3h + 1} + 1}$

Theorem 3.1 (Extensions to Power and Radical Rules)**.** *Let* $\lim_{x \to c} f(x) = L$. *Then*

(7') Power

$$\lim_{x \to c} [f(x)]^n = L^n.$$

(8') Radical

$$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L}.$$

Example. Evaluate the following limits.

(1) $\lim_{x \to 2} \sqrt{x^2 + 2x - 4}$

(2) $\lim_{x \to 1} (x^3 + 2x^2 + x - 3)^5$

Theorem 3.2 (Limits of Transcendental Functions)**.** *Let c be a number in the domain of the given transcendental function.*

(1) $\lim_{x \to c} \sin x = \sin c$.

(2) $\lim_{x \to c} \cos x = \cos c$.

(3) $\lim_{x \to c} \tan x = \tan c$.

(4) $\lim_{x \to c} \cot x = \cot c$.

(5) $\lim_{x \to c} \sec x = \sec c$.

(6) $\lim_{x \to c} \csc x = \csc c$.

(7) $\lim_{x \to c} a^x = a^c,\ a > 0$.

(8) $\lim_{x \to c} \ln x = \ln c$.

Example. Evaluate the following limits.

(1) $\lim_{x \to 0} \sin x$

(2) $\lim_{x \to \pi} \sec x$

(3) $\lim_{x \to -\pi/4} \sec x \tan x$

(4) $\lim_{x \to 0} \ln(\sec x)$

(5) $\lim_{x \to 3} 2^{x-1}$

(6) $\lim_{x \to \pi} e^{\sin x}$

Theorem 3.3 (Functions That Agree at All but One Point)**.** *Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as $x \to c$ exists, then the limit of $f(x)$ also exists and*

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$

Definition (Indeterminate Form)**.** An **indeterminate form** is a meaningless numerical expression. Examples include:

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty \qquad 0^0 \qquad 1^\infty \qquad \infty^0$$

Example. Evaluate the following limits.

(1) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x-1}$

(2) $\displaystyle\lim_{u\to 1} \frac{u^4-1}{u^3-1}$

(3) $\displaystyle\lim_{x\to -1} \frac{x^2+3x+2}{x^2-x-2}$

(4) $\displaystyle\lim_{x\to 2} \frac{x^2-7x+10}{x-2}$

(5) $\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$

(6) $\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1}$

(7) $\lim_{x \to 1} \frac{x - 1}{\sqrt{x + 3} - 2}$

(8) $\lim_{x \to -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$

Infinite Limits

When evaluating limits of rational functions, one of three cases occur:

(1) The limit returns a number, possibly 0. The value is the limit.

(2) The limit returns the indeterminate form $0/0$. Some algebraic rearrangement needs to be done to evaluate the limit, like above.

(3) The limit returns $\dfrac{\text{nonzero}}{0}$. In this case, the limit can be $+\infty$ or $-\infty$, or the limit may not exist. The result depends on the values of the left-hand and right-hand limits.

Example. Evaluate the following limits.

(1) $\displaystyle\lim_{x\to 1} \frac{x^2-1}{2x+4}$

(2) $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2-4}$

(3) $\displaystyle\lim_{x\to 2} \frac{(x-2)^2}{x^2-4}$

(4) $\displaystyle\lim_{x\to 2} \frac{x-3}{x^2-4}$

(5) $\lim \dfrac{x^2 + x - 6}{x^3 - 2x^2}$

(a) $x \to 0$

(b) $x \to 2$

(6) $\lim\limits_{x \to 0} \dfrac{3}{x^{2/5}}$

(7) $\lim \dfrac{x^2 - 3x + 2}{x^3 - 4x}$

(a) $x \to -2$

(b) $x \to 0$

(c) $x \to 1$

(d) $x \to 2$

(8) $\lim_{x \to \pi/2^-} \tan x$

(9) $\lim_{x \to \pi^+} \csc x$

(10) $\lim_{x \to \pi/2} \sec x$

(11) $\lim_{x \to 0^+} \ln x$

Piecewise Functions

Example. Evaluate the following limits.

(1) Let

$$f(x) = \begin{cases} 3 - x & x < 3 \\ 2 & x = 2 \\ \frac{x}{2} & x > 2 \end{cases}$$

Find $\lim_{x \to 2} f(x)$ and $\lim_{x \to -1} f(x)$.

(2) Let

$$f(x) = \begin{cases} x + 2 & x < 0 \\ e^x & 0 \leq x \leq 1 \\ 2 - x & x > 1 \end{cases}$$

Find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$.