30 Wednesday, November 15

Review

Definition.

(1) An *antiderivative* of a function f is a function F such that $F' = f$. In general, f has infinitely many antiderivatives, all of them differing by an additive constant, i.e., $F - G = C$ a constant for two antiderivatives F and G of f.

(2) The *definite integral* of a function f from $x = a$ to $x = b$, represented as

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(c_i)\Delta x$$

is the limit of a Riemann sum as the number of rectangles increases without bound. It is the measure of the (signed) area under the graph of $f(x)$ from $x = a$ to $x = b$. Inside of the definite integral, the variable of integration is a *dummy variable*. It is merely a place holder.

The Fundamental Theorem of Calculus

Consider the following function. Construct the definite integral of a function $f(t)$ from $t = c$ to $t = x$, where the lower bound is fixed at a constant c and the upper bound is allowed to vary.

$$F(x) = \int_c^x f(t)\,dt$$

If x is allowed to increase by some small amount Δx, then

Theorem.

(1) If $f(x)$ is a continuous function on $[a, b]$, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

(2) If $f(x)$ is a continuous function on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Example. Evaluate the following definite integrals.

(1) $\displaystyle\int_{-1}^{2} \left(x^3 - 2x\right) dx$

(2) $\displaystyle\int_{1}^{4} \left(5 - 2t + 3t^2\right) dt$

(3) $\displaystyle\int_{1}^{9} \sqrt{x}\, dx$

(4) $\displaystyle\int_{\pi/6}^{\pi} \sin\theta\, d\theta$

(5) $\displaystyle\int_1^9 \frac{2x-5}{\sqrt{x}}\,dx$

(6) $\displaystyle\int_1^{18} \sqrt{\frac{3}{x}}\,dz$

(7) $\displaystyle\int_{\pi/4}^{\pi/3} \csc^2\theta\,d\theta$

(8) $\displaystyle\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta}\,d\theta$

(9) $\int_{-5}^{5} e \, dx$

(10) $\int_{-1}^{1} e^{u+1} \, du$

Example. Find the area of the region between the given set of curves.

(1) $y = \sqrt[3]{x}, \quad y = 0, \quad x = 0, \quad x = 27$

(2) $y = x^{-4}, \quad y = 0, \quad x = 1, \quad x = 6$

(3) $y = \sin x, \quad y = 0, \quad x = 0, \quad x = \pi$

(4) $y = \sec^2 x, \quad y = 0, \quad x = 0, \quad x = \pi/3$