33 Wednesday, November 29

Review: Differential Equations

A **differential equation** is an equation that relates a function to its derivatives. The simplest example is an equation like

$$\frac{dy}{dt} = f(t)$$

where the right hand side involves only t. Another example is

$$\frac{dy}{dt} = ky$$

Qualitatively, this equation says, "the rate of change of y is directly proportional to y itself." To solve:

Theorem (Law of Exponential Growth/Decay)**.** If y is a differentiable function of t such that $y > 0$ and $y' = ky$ for some constant k, then

$$y = Ce^{kt}$$

where C is the **initial value** of y (the value of y at $t = 0$) and k is the **proportionality constant**. If $k > 0$, then y is growing exponentially. If $k < 0$, then y is decaying exponentially.

Example. Suppose y is a function such that its rate of change is directly proportional to itself. Find y if:

(1) when $t = 0$, $y = 10$ and $t = 1$, $y = 12$

(2) when $t = 0$, $y = 1000$ and $t = 1$, $y = 100$

(3) when $t = 0$, $y = 20$, and $t = 5$, $y = 1000$

Example. Solve the differential equations.

(1) $\dfrac{dy}{dt} = 6y, y(0) = 10$

(2) $\dfrac{dy}{dt} = y \ln 4, y(1) = 1/2$

Real-world models that involve exponential growth/decay:

(1) Population growth

(2) Radioactive decay

(3) Interest

Example.

(1) Suppose that a colony of bacteria obeys the law of exponential growth, and suppose that a colony started with 1 bacteria and every hour the population doubles. What is the population after 24 hours?

(2) The population of a country is growing exponentially. 3 years from now, the population will be 10 million, and 5 years from now it will be 40 million. How many people are in the country today?

Interest

If an amount A is invested at an annual interest rate r and is compounded k times per year (that is, interest is added k times per year), then after t years,

$$A(t) = A\left(1 + \frac{r}{k}\right)^{kt}$$

Interest may be compounded monthly ($k = 12$), weekly ($k = 52$), daily ($k = 365$), or even more frequently. Taking the limit gives

$$\lim_{k \to \infty} A(t) = \lim_{k \to \infty} A\left(1 + \frac{r}{k}\right)^{kt} = Ae^{rt}$$

This is the result of interest **compounded continuously**.

Example.

(1) Suppose you deposit \$621 in a bank account that pays 6% annual interest compounded continuously. How much money will you have 8 years later? How long does it take to double your investment?

(2) Suppose you deposit \$1000 in a bank account that pays 4% annual interest compounded continuously. How much money will you have 5 years later? How long does it take to double your investment?

(3) Suppose you have \$500 to invest, and you want to double your investment in 6 years. Assuming continuous compounding, what annual interest rate would be required to do so?