# 4 Wednesday, August 30

## Continuity

**Definition** (Continuity)**.** A function $f$ is **continuous at** $x = c$ if

(i) $f(c)$ exists.

(ii) $\lim_{x\to c} f(x)$ exists.

(iii) $\lim_{x\to c} f(x) = f(c)$.

Otherwise, $f$ is **discontinuous at** $x = c$. Furthermore, if $f$ is continuous at every $x$ in the open interval $(a, b)$, then $f$ is continuous on $(a, b)$. If $f$ is continuous on $(a, b)$ and

$$\lim_{x\to a^+} f(x) = f(a) \qquad \lim_{x\to b^-} f(x) = f(b),$$

then $f$ is continuous on $[a, b]$.

**Note.** Graphically, this definition means that $f$ is continuous on an interval if and only if the graph of $f$ can be drawn with a single, unbreaking stroke.

**Example.** Examples of continuous functions:

(1) Polynomials are continuous everywhere.

$$f(x) = x + 1 \qquad g(x) = x^2 + 3 \qquad h(x) = x^{100} - x$$

(2) Rational functions are continuous on their domains.

$$f(x) = \frac{x+1}{x-2} \qquad (-\infty, 2) \cup (2, \infty)$$

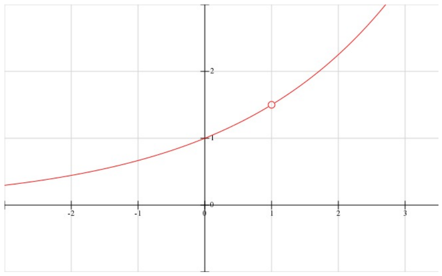
$$g(x) = \frac{x}{x^2 + 3x + 2} \qquad (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$$

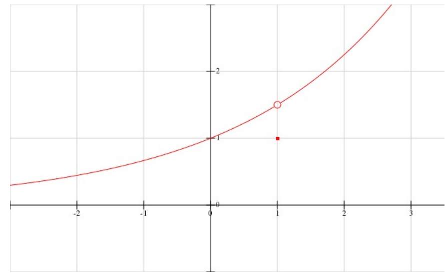
(3) If $f$ and $g$ are continuous at $x = c$, then $kf$ ($k$ a real number), $f \pm g$, $fg$, and $\frac{f}{g}$ ($g(c) \neq 0$) are continuous at $x = c$.

(4) Trigonometric, exponential, and logarithmic functions are all continuous everywhere on their domain.

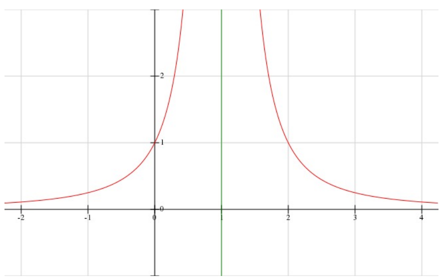
**Example.** Examples of discontinuous functions:

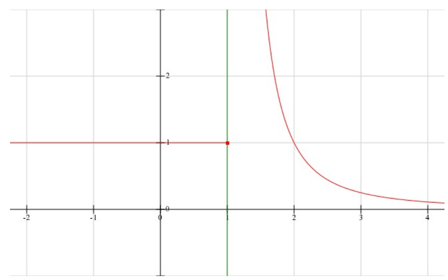
(1) Gap/hole discontinuities



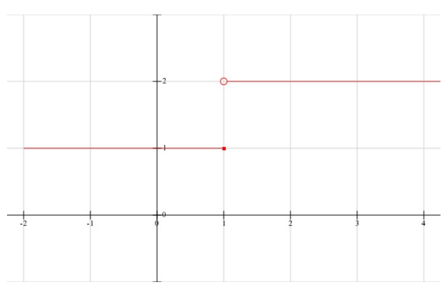


(2) Infinite discontinuities





(3) Jump discontinuities



## Continuity of Piecewise Functions

To find where a piecewise function is continuous:

(1) Check where each branch is continuous as a function on its own. Compare this with the interval that the branch is defined, taking only the relevant part.

(2) Use the definition of continuity to check the continuity of points where the branches change.

**Example.** Find the points of discontinuity for each of the following functions. Are the function continuous from the left or right at these points?

(1) $f(x) = \begin{cases} x^2 - 3x + 1 & x \neq 3 \\ 2 & x = 3 \end{cases}$

(2) $f(x) = \begin{cases} 4x + 5 & x \leq -1 \\ x^2 + 1 & x > -1 \end{cases}$

$$(3)\quad f(x) = \begin{cases} x^2 - 1 & -1 \le x \le 0 \\ 2x & 0 < x < 1 \\ 1 & x = 1 \\ -2x + 4 & 1 < x < 2 \\ 0 & 2 < x < 3 \end{cases}$$

$$(4)\quad f(x) = \begin{cases} 1 + x^2 & x \le 0 \\ 2 - x & 0 < x \le 2 \\ (x-2)^2 & x > 2 \end{cases}$$

$$(5)\quad f(x) = \begin{cases} x + 1 & x \le 1 \\ 1/x & 1 < x < 3 \\ \sqrt{x-3} & x \ge 3 \end{cases}$$

**Example.** For each of the following functions, find the values of the unknown variables that make each function continuous everywhere.

(1) $f(x) = \begin{cases} ax^2 + 2x & x < 2 \\ x^3 - ax & x \geq 2 \end{cases}$

(2) $h(x) = \begin{cases} x^2 + a\cos x + 4 & x \leq 0 \\ x + 3 & x > 0 \end{cases}$

(3) $g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x < 2 \\ ax^2 - bx + 3 & 2 \leq x < 3 \\ 2x - a + b & x \geq 3 \end{cases}$

**Theorem 4.1** (Intermediate Value Property)**.** *If $f$ is continuous on $[a, b]$ and $L$ is such that $f(a) \leq L \leq f(b)$, then there exists a number $c$, $a \leq c \leq b$, such that $f(c) = L$.*